HYPERBOLIC TESSELLATIONS AND GENERATORS OF K_3 FOR IMAGINARY QUADRATIC FIELDS

DAVID BURNS, ROB DE JEU, HERBERT GANGL, ALEXANDER D. RAHM, AND DAN YASAKI

Abstract. We develop methods for constructing explicit generating elements, modulo torsion, of the K_3-groups of imaginary quadratic number fields. These methods are based on either tessellations of hyperbolic 3-space or on direct calculations in suitable pre-Bloch groups, and lead to the very first proven examples of explicit generators, modulo torsion, of any infinite K_3-group of a number field. As part of this approach, we make several improvements to the theory of Bloch groups for K_3 of any infinite field, predict the precise power of 2 that should occur in the Lichtenbaum conjecture at -1, and prove that this prediction is valid for all abelian number fields.

Contents

2010 *Mathematics Subject Classification.* Primary: 11R70, 19F27, 19D45; secondary: 11R42, 20H20, 51M20.
Key words and phrases. regulator, Lichtenbaum conjecture, Bloch group, imaginary quadratic field, hyperbolic 3-space, tessellations, Bianchi group.

1. Introduction

1.1. **The general context.** Let F be a number field with ring of algebraic integers $\mathcal{O}_F$. Then, for each natural number m with $m > 1$, the algebraic K-group $K_m(\mathcal{O}_F)$ in degree m of Quillen is a fundamental invariant of F, constituting a natural generalization in even degrees of the ideal class group of $\mathcal{O}_F$ and in odd degrees of the group of units of $\mathcal{O}_F$.

Following fundamental work of Quillen [38], and of Borel [6], the abelian groups $K_m(\mathcal{O}_F)$ are known to be finite in positive even degrees and finitely generated in all degrees.

In addition, as a natural generalization of the analytic class number formula, Lichtenbaum [36] has conjectured that the leading coefficient $\zeta_F^*(1-m)$ in the Taylor expansion at $s = 1-m$ of the Riemann zeta function $\zeta_F(s)$ of F should satisfy

$$\zeta_F^*(1-m) = \pm 2^{n_{m,F}} \frac{|K_{2m-2}(\mathcal{O}_F)|}{|K_{2m-1}(\mathcal{O}_F)_{\rm tor}|} R_m(F). \tag{1.1}$$

Here we write $|X|$ for the cardinality of a finite set X, $K_{2m-1}(\mathcal{O}_F)_{\rm tor}$ for the torsion subgroup of $K_{2m-1}(\mathcal{O}_F)$, $R_m(F)$ for the covolume of the image of $K_{2m-1}(\mathcal{O}_F)$ under the Beilinson regulator map, and $n_{m,F}$ for an undetermined integer.

However, whilst Borel [7] has proved that the quotient of $\zeta_F^*(1-m)$ by $R_m(F)$ is always rational (see Theorem 2.1), the only family of fields F for which (1.1) is known to be unconditionally valid are abelian extensions of $\mathbb{Q}$ (cf. Remark 2.10).

In addition, it has remained a difficult problem to explicitly compute, except in special cases, either $|K_{2m-2}(\mathcal{O}_F)|$ or $R_m(F)$, or to describe explicit generators of $K_{2m-1}(\mathcal{O}_F)$ modulo torsion.

These are then the main problems that motivated the present article. To address them we are led to clarify the integer $n_{m,F}$ that should occur in (1.1), to investigate the fine integral properties of various Bloch groups, to develop techniques for checking divisibility in such groups and, in the case of imaginary quadratic fields F (which are the number fields of lowest degree for which the group $K_3(\mathcal{O}_F)$ is infinite), to use ideal tessellations of hyperbolic 3-space to explicitly construct elements in a suitable Bloch group that can be shown to generate a subgroup of index $|K_2(\mathcal{O}_F)|$ in the quotient group $K_3(\mathcal{O}_F)/K_3(\mathcal{O}_F)_{\rm tor}$.

Whilst each of these aspects is perhaps of independent interest, it is only by understanding the precise interplay between them that we are able to make significant progress on the above problems.

In fact, as far as we are aware, our study is the first in which the links between these theories have been investigated in any precise, or systematic, way in the setting of integral (rather than rational) coefficients, and it seems likely that further analysis in this direction can lead to additional insights.

At this stage, at least, we have used the approach developed here to explicitly determine for all imaginary quadratic fields k of absolute discriminant at most 1000 a generator, modulo torsion, of $K_3(\mathcal{O}_k)$, the order of $K_2(\mathcal{O}_k)$ and the value of the Beilinson regulator $R_2(k)$.

In this way we have obtained the very first proven examples of explicit generators, modulo torsion, of the K_3-group of any number field for which the group is infinite, thereby resolving a problem that has been considered ever since K_3-groups were introduced. We have also determined the order of $K_2(\mathcal{O}_k)$ in several interesting new cases.

These computations both rely on, and complement, earlier work of Belabas and the third author in [2] concerning the orders of such $K_2(\mathcal{O}_k)$. In particular, our computations show that the (divisional) bounds on $|K_2(\mathcal{O}_k)|$ that are obtained in loc. cit. (in those cases where the order could not be precisely established) are sharp, and also prove an earlier associated conjecture of Browkin and the third author from [11].

In a different direction, our methods have also given the first concrete evidence to suggest both that the groups defined by Suslin (in terms of group homology) in [43] and by Bloch (in terms of relative K-theory) in [5] should be related in a very natural way, and also that Bloch's group should account for all of the indecomposable K_3-group, at least modulo torsion.

In addition, some of the techniques developed here extend to number fields F that need not be abelian (or even Galois) over $\mathbb{Q}$, in which case essentially nothing of a general nature beyond the result of Borel is known about Lichtenbaum's conjectural formula (1.1).

For such fields our methods can in principle be used to either construct elements that generate a finite index subgroup of $K_3(\mathcal{O}_F)$ and so can be used to investigate the possible validity of (1.1) or, if the more precise form of (1.1) discussed below is known to be valid for F and $m = 2$, to construct a full set of generators, modulo torsion, of $K_3(\mathcal{O}_F)$. For the sake of brevity, however, we shall not pursue these aspects in the present article.

1.2. **The main results.** For the reader's convenience, we shall now discuss in a little more detail the main contents of this article.

In §2 we address the issue of the undetermined exponent in (1.1) by proving that the Tamagawa number conjecture that was formulated originally by Bloch and Kato in [4], and then subsequently extended by Fontaine and Perrin-Riou in [22], predicts a precise, and more-or-less explicit, formula for $n_{m,F}$. In the special case that $m = 2$ we can in fact make this formula completely explicit (though, in general, still conjectural) by using a result of Levine from [35]. In addition, by using results of Huber and Kings [26], of Greither and the first author [15] and of Flach [21] relating to the Tamagawa number conjecture we are then able to prove the (unconditional) validity of (1.1), with a precise form of the exponent $n_{m,F}$, for all m and for all fields F that are abelian over $\mathbb{Q}$.

The main result of §2 plays a key role in our subsequent computations and is also we feel of some independent interest. However, the arguments in §2 are technical in nature, relying on certain 2-adic constructions and results of Kahn [28] and Rognes and Weibel [39] and on detailed computations of the determinant modules of Galois cohomology complexes. Because these methods are not used elsewhere in the article we invite any reader whose main interest is the determination of explicit generators of K_3-groups to simply read this section up to the end of §2.1 and then pass on to §3.

In §3 we shall then introduce a useful modification $\overline{\mathfrak{p}}(F)$ of the pre-Bloch group $\mathfrak{p}(F)$ that is defined by Suslin in [43] (for in fact any infinite field), and explain how this relates both to Suslin's construction and to (degenerate) configurations of points.

We shall also explicitly compute the torsion subgroup of the resulting modified Bloch group $\overline{B}(F)$ for a number field F and of the corresponding variant of the second exterior power of F^* in a way that is useful for implementation purposes. In particular, we show that $\overline{B}(F)$ is torsion-free if F is an imaginary quadratic field and hence much more amenable to our computational approach than is Suslin's original construction.

In this section we shall also use a result of the first three authors in [16] to construct a canonical homomorphism ψ_F from $\overline{B}(F)$ to $K_3(F)^{\mathrm{ind}}_{\mathrm{tf}}$, the indecomposable K_3-group of F modulo torsion. In the case that F is a number field, we are led, on the basis of extensive numerical evidence, to conjecture that ψ_F is bijective. This gives new insight into the long-standing questions concerning the precise connection between Suslin's construction and the earlier construction of Bloch in [5] and whether Bloch's construction accounts for all of the group $K_3(F)^{\mathrm{ind}}_{\mathrm{tf}}$.

In this regard, it is also of interest to note that our approach to relating configurations of points to the pre-Bloch group differs slightly, but crucially, from that described by Goncharov in [24, §3]. In particular, by these means we are able to make computations without having to ignore torsion of exponent two, as is necessary in the approach of Goncharov. (We remark that this ostensibly minor improvement is in fact essential to our approach to compute explicit generating elements of K_3-groups.)

In §4-6 we specialise to consider the case of an imaginary quadratic field k.

In this case we shall firstly, in §4, invoke a tessellation of hyperbolic 3-space $\mathbb{H}^3$, based on perfect forms, to construct an explicit element β_{geo} of the group $\overline{B}(k)$ defined in §3.

The polyhedral reduction theory for $\mathrm{GL}_2(\mathcal{O}_k)$ that has been developed by Ash [1, Chap. II] and Koecher [31] plays a key role in this construction, and Humbert's classical formula for the value $\zeta_k(2)$ in terms of the volume of a fundamental domain for the action of $\mathrm{PGL}_2(\mathcal{O}_k)$ on $\mathbb{H}^3$ allows us to explicitly describe the image under the Beilinson regulator map of $\psi_k(\beta_{\mathrm{geo}})$ in terms of the leading term $\zeta_k^*(-1)$. By using the known validity of a completely precise form of the equality (1.1) for $F = k$ and $m = 2$ we are then able to deduce that $\psi_k(\beta_{\mathrm{geo}})$ generates a subgroup of $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$ of index $|K_2(\mathcal{O}_k)|$.

The proof that our construction of β_{geo} gives a well-defined element of $\overline{B}(k)$ is lengthy, and rather technical, since it relies on a detailed study of the polytopes that arise in the tessellation constructed in §4 and, for this reason, it is deferred to §5.

Then in §6 we shall combine results from previous sections to describe two concrete approaches to finding explicit generators of $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$ and the order of $K_2(\mathcal{O}_k)$ for many imaginary quadratic number fields k.

The first approach is discussed in §6.1 and depends upon dividing the element $\psi_k(\beta_{\mathrm{geo}})$ constructed in §4 by $|K_2(\mathcal{O}_k)|$ in an algebraic way in order to obtain a generator of $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$. This method partly relies on constructing non-trivial elements in $\overline{B}(k)$ by means of an algorithmic process involving exceptional S-units that is described in §6.3.

The second approach is discussed in §6.2 and again relies on the algorithm described in §6.3 to generate elements in $\overline{B}(k)$, and hence, via the map ψ_k, in $K_3(k)^{\text{ind}}_{\text{tf}}$ and upon knowing the validity of a completely precise form of (1.1). In this case, however, we combine these aspects with some (in practice sharp) bounds on $|K_2(\mathcal{O}_k)|$ that are provided by [2] in order to draw algebraic conclusions from numerical calculations, leading us to the computation of a generator of $K_3(k)^{\text{ind}}_{\text{tf}}$ (or of $\overline{B}(k)$) as well as of $|K_2(\mathcal{O}_k)|$ in many interesting cases. As a concrete example, we describe an explicit result for an imaginary quadratic field k for which we found that $|K_2(\mathcal{O}_k)|$ is equal to the prime 233 (thereby verifying a conjecture from [11]).

The article then concludes with two appendices. In Appendix A we shall prove several useful results about finite subgroups of $\mathrm{PGL}_2(\mathcal{O}_k)$ of an imaginary quadratic field k that are needed in earlier arguments, but for which we could not find a suitable reference. Finally, in Appendix B we shall give details of the results of applying the geometrical construction of §4 and the approach described in §6.1 to an imaginary quadratic field k for which $|K_2(\mathcal{O}_k)|$ is equal to 22.

1.3. **Notations and conventions.** As a general convention we use $\mathcal{F}$ to denote an arbitrary field (assumed in places to be infinite), F to denote a number field and k to denote an imaginary quadratic field (or, rarely, $\mathbb{Q}$).

For a number field F we write $\mathcal{O}_F$ for its ring of integers, D_F for its discriminant, and $r_1(F)$ and $r_2(F)$ for the number of its real and complex places respectively.

For an imaginary quadratic field k we set

$$\omega = \omega_k := \begin{cases} \sqrt{D_k/4} & \text{if } D_k \equiv 0 \bmod 4, \\ (1+\sqrt{D_k})/2 & \text{if } D_k \equiv 1 \bmod 4 \end{cases}$$

so that $k = \mathbb{Q}(\omega)$ and $\mathcal{O}_k = \mathbb{Z}[\omega]$.

For an abelian group M we write M_{tor} for its torsion subgroup and M_{tf} for the quotient group M/M_{tor}. The cardinality of a finite set X will be denoted by $|X|$.

1.4. **Acknowledgements.** We are grateful to the Irish Research Council for funding a work stay of the fourth author in Amsterdam, and to the De Brun Centre for Computational Homological Algebra for funding a work stay of the second author in Galway. The initial research for this paper was strongly propelled by these two meetings of the authors.

We also thank Mathematisches Forschungsinstitut Oberwolfach for hosting the last four authors to continue this work as part of their Research in Pairs program.

We would like to thank Philippe Elbaz-Vincent for useful conversations and Nguyen Quang Do for helpful comments.

The fifth author was partially supported by NSA grant H98230-15-1-0228. This manuscript is submitted for publication with the understanding that the United States government is authorized to produce and distribute reprints.

2. The conjectures of Lichtenbaum and of Bloch and Kato

It has long been known that the validity of (1.1) follows from that of the conjecture originally formulated by Bloch and Kato in [4] and then reformulated and extended by Fontaine in [23] and by Fontaine and Perrin-Riou in [22] (see Remark 2.8 below).

However, for the main purpose of this article, it is essential that one knows not just the validity of (1.1) but also an explicit value of the exponent $n_{m,F}$ for the number field F.

In this section we shall therefore derive an essentially precise formula for $n_{m,F}$ from the assumed validity of the above conjecture of Bloch and Kato.

If M is a $\mathbb{Z}_p$-module, then we identify $M_{\rm tf}$ with its image in $\mathbb{Q}_p \cdot M := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$ and for any homomorphism of $\mathbb{Z}_p$-modules $\theta : M \to N$ we write $\theta_{\rm tf}$ for the induced homomorphism $M_{\rm tf} \to N_{\rm tf}$.

We write $D(\mathbb{Z}_2)$ for the derived category of $\mathbb{Z}_2$-modules and $D^{\rm perf}(\mathbb{Z}_2)$ for the full triangulated subcategory of $D(\mathbb{Z}_2)$ comprising complexes that are isomorphic (in $D(\mathbb{Z}_2)$) to a bounded complex of finitely generated modules.

2.1. **Statement of the main result.** Throughout this section, F denotes a number field.

2.1.1. We first review Borel's Theorem.

For each subring Λ of $\mathbb{R}$ and each integer a, we write $\Lambda(a)$ for the subset $(2\pi i)^a \cdot \Lambda$ of $\mathbb{C}$.

We then fix a natural number m and recall that Beilinson's regulator map

$$\mathrm{reg}_m : K_{2m-1}(\mathbb{C}) \to \mathbb{R}(m-1)$$

is compatible with the natural actions of complex conjugation on $K_{2m-1}(\mathbb{C})$ and $\mathbb{R}(m-1)$.

For each embedding $\sigma : F \to \mathbb{C}$, we consider the composite homomorphism

$$\mathrm{reg}_{m,\sigma} : K_{2m-1}(\mathcal{O}_F) \xrightarrow{\sigma_*} K_{2m-1}(\mathbb{C}) \xrightarrow{\mathrm{reg}_m} \mathbb{R}(m-1)$$

where σ_* denotes the induced map on K-groups.

We also write $\zeta_F^*(1-m)$ for the first non-zero coefficient in the Taylor expansion at $s = 1-m$ of the Riemann zeta function $\zeta_F(s)$ of F and define a natural number

$$d_m(F) := \begin{cases} r_2(F), & \text{if } m \text{ is even,} \\ r_1(F) + r_2(F), & \text{if } m \text{ is odd.} \end{cases}$$

Theorem 2.1 (Borel's theorem). *For each natural number m with $m > 1$ the following claims are valid.*

(i) The rank of $K_{2m-2}(\mathcal{O}_F)$ is zero.
(ii) The rank of $K_{2m-1}(\mathcal{O}_F)$ is $d_m(F)$.
(iii) Write $\mathrm{reg}_{m,F}$ for the map $K_{2m-1}(\mathcal{O}_F) \to \prod_{\sigma:F\to\mathbb{C}} \mathbb{R}(m-1)$ given by $(\mathrm{reg}_{m,\sigma})_\sigma$. Then the image of $\mathrm{reg}_{m,F}$ is a lattice in the real vector space $V_m = \{(c_\sigma)_\sigma | c_{\overline{\sigma}} = \overline{c_\sigma}\}$, and its kernel is $K_{2m-1}(\mathcal{O}_F)_{\rm tor}$.
(iv) Normalize covolumes in V_m so that the lattice $\{(c_\sigma)_\sigma | c_{\overline{\sigma}} = \overline{c_\sigma}$ and $c_\sigma \in \mathbb{Z}(m-1)\}$ has covolume 1, and let $R_m(F)$ be the covolume of $\mathrm{reg}_{m,F}$. Then $\zeta_F(s)$ vanishes to order $d_m(F)$ at $s = 1-m$, and there exists a non-zero rational number $q_{m,F}$ so that $\zeta_F^*(1-m) = q_{m,k} \cdot R_m(F)$.

2.1.2. For each pair of integers i and j with $j \in \{1,2\}$ and $i \geq j$ and each prime p there exist canonical 'Chern class' homomorphisms of $\mathbb{Z}_p$-modules

$$K_{2i-j}(\mathcal{O}_F) \otimes \mathbb{Z}_p \to H^j(\mathcal{O}_F[1/p], \mathbb{Z}_p(i)). \tag{2.2}$$

The first such homomorphism $\mathrm{c}^{\rm S}_{F,i,j,p}$ was constructed using higher Chern class maps by Soulé in [41] (with additional details in the case $p = 2$ being provided by Weibel in [49]), a second

$\mathrm{c}^{\mathrm{DF}}_{F,i,j,p}$ was constructed using étale K-theory by Dwyer-Friedlander in [19] and, in the case $p = 2$, there is a third variant, introduced independently by Kahn [28] and by Rognes and Weibel [39], that will play a key role in later arguments. In each case the maps are natural in $\mathcal{O}_F$ and are known to have finite kernels and cokernels (see the discussion in §2.6 for the case $p = 2$) and hence to induce isomorphisms of the associated $\mathbb{Q}_p$-spaces.

Since the prime $p = 2$ will be of most interest to us we usually abbreviate the maps $\mathrm{c}^{\mathrm{S}}_{F,i,j,2}$ and $\mathrm{c}^{\mathrm{DF}}_{F,i,j,2}$ to $\mathrm{c}^{\mathrm{S}}_{F,i,j}$ and $\mathrm{c}^{\mathrm{DF}}_{F,i,j}$ respectively.

We can now state the main result of this section concerning the undetermined rational number $q_{m,F}$ that occurs in Theorem 2.1(iv).

Theorem 2.3. *Fix an integer m with $m > 1$. Then the Bloch-Kato Conjecture is valid for the motive $h^0(\mathrm{Spec}(F))(1-m)$ if and only if one has*

$$q_{m,F} = (-1)^{s_m(F)} 2^{r_2(F)+t_m(F)} \cdot \frac{|K_{2m-2}(\mathcal{O}_F)|}{|K_{2m-1}(\mathcal{O}_F)_{\mathrm{tor}}|}. \tag{2.4}$$

Here we set

$$s_m(F) := \begin{cases} [F:\mathbb{Q}]\frac{m}{2} - r_2(F), & \text{if } m \text{ is even,} \\ [F:\mathbb{Q}]\frac{m-1}{2}, & \text{if } m \text{ is odd,} \end{cases}$$

and $t_m(F) := r_1(F) \cdot t^1_m(F) + t^2_m(F)$ with

$$t^1_m(F) := \begin{cases} -1, & \text{if } m \equiv 1 \ (\text{mod } 4) \\ -2, & \text{if } m \equiv 3 \ (\text{mod } 4) \\ 1, & \text{otherwise,} \end{cases}$$

and $t^2_m(F)$ the integer which satisfies

$$2^{t^2_m(F)} := |\mathrm{cok}(\mathrm{c}^{\mathrm{S}}_{F,m,1,\mathrm{tf}})| \cdot 2^{-a_m(F)} \equiv \det_{\mathbb{Q}_2}((\mathbb{Q}_2 \cdot \mathrm{c}^{\mathrm{S}}_{F,m,1}) \circ (\mathbb{Q}_2 \cdot \mathrm{c}^{\mathrm{DF}}_{F,m,1})^{-1}) \pmod{\mathbb{Z}_2^\times} \tag{2.5}$$

where $a_m(F)$ is 0 except possibly when both $m \equiv 3$ (mod 4) and $r_1(F) > 0$ in which case it is an integer satisfying $0 \le a_m(F) < r_1(F)$.

Remark 2.6. The main result of Burgos Gil's book [12] implies that the m-th Borel regulator of F is equal to $2^{d_m(F)} \cdot R_m(F)$ and so (2.4) leads directly to a more precise form of the conjectural formula for $\zeta^*_F(1-m)$ in terms of Borel's regulator that is given by Lichtenbaum in [36].

Remark 2.7. The proof of Lemma 2.25(ii) below gives a closed formula for the integer $a_m(F)$ that occurs in Theorem 2.3 (and see also Remark 2.26 in this regard). In addition, in [35, Th. 4.5] Levine shows that $\mathrm{c}^{\mathrm{S}}_{F,2,1}$, and hence also $\mathrm{c}^{\mathrm{S}}_{F,2,1,\mathrm{tf}}$, is surjective. It follows that $t^2_2(F) = 0$ and hence that the exponent of 2 in the expression (2.4) is completely explicit in the case $m = 2$. In general, it is important for the sort of numerical computations that we make subsequently to give an explicit upper bound on $t^2_m(F)$, and hence also on $t_m(F)$, and such a bound follows directly from Theorem 2.27 below.

Remark 2.8. Up to sign and an undetermined power of 2, the result of Theorem 2.3 is proved by Huber and Kings in [26, Th. 1.4.1] under the assumption that $\mathrm{c}^{\mathrm{S}}_{F,m,j,p}$ is bijective for all odd primes p, as is conjectured by Quillen and Lichtenbaum. In addition, Suslin has shown the latter conjecture is a consequence of the conjecture of Bloch and Kato relating Milnor K-theory to étale cohomology and, following fundamental work of Voevodsky and Rost, Weibel completed the proof of the Bloch-Kato Conjecture in [48]. Our contribution to the proof of Theorem 2.3 is thus concerned solely with specifying the sign (which is straightforward) and the precise power of 2 that should occur.

If F is an abelian field, then the Bloch-Kato conjecture for $h^0(\mathrm{Spec}(F))(1-m)$ is known to be valid: up to the 2-primary part, this was verified indepedendently by Huber and Kings in [26] and by Greither and the first author in [15] and the 2-primary component was subsequently resolved by Flach in [21]. Theorem 2.3 thus leads directly to the following result.

Corollary 2.9. *The formula (2.4) is unconditionally valid if F is an abelian field.*

Remark 2.10. Independently of connections to the Bloch-Kato conjecture, the validity of (1.1), but not (2.4), for abelian fields F was first established by Kolster, Nguyen Quang Do and Fleckinger in [32] (note, however, that the main result of loc. cit. contains certain erroneous Euler factors and that the necessary correction is provided by Benois and Nguyen Quang Do in [3, §A.3]). The general approach of [32] also provided motivation for the subsequent work of Huber and Kings in [26].

Remark 2.11. In the special case that F is equal to an imaginary quadratic field k, Corollary 2.9 combines with Theorem 2.1(iv), Remark 2.7 and Example 3.1 below to imply that the equality

$$\zeta_k^*(-1) = -2\frac{|K_2(\mathcal{O}_k)|}{24}R_2(k) \tag{2.12}$$

is unconditionally valid.

2.2. **The proof of Theorem 2.3: a first reduction.** In the sequel we abbreviate the non-negative integers $r_1(F)$, $r_2(F)$ and $d_m(F)$ to r_1, r_2 and d_m respectively.

The functional equation of $\zeta_F(s)$ then has the form

$$\zeta_F(1-s) = \frac{2^{r_2}\cdot \pi^{[F:\mathbb{Q}]/2}}{|D_F|^{1/2}}\left(\frac{|D_F|}{\pi^{[F:\mathbb{Q}]}}\right)^s\left(\frac{\Gamma(s)}{\Gamma(1-s)}\right)^{r_2}\left(\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\right)^{r_1}\cdot\zeta_F(s). \tag{2.13}$$

where $\Gamma(s)$ is the Gamma function.

Since m is in the region of convergence of $\zeta_F(s)$ the value $\zeta_F(m)$ is a strictly positive real number. In addition, the function $\Gamma(s)$ is analytic and strictly positive for $s>0$, is analytic, non-zero and of sign $(-1)^{\frac{1}{2}-n}$ at each strictly negative half-integer n and has a simple pole at each strictly negative integer n with residue of sign $(-1)^n$.

Given these facts, the above functional equation implies directly that $\zeta_F(s)$ vanishes at $s = 1-m$ to order d_m (as claimed in Theorem 2.1(iv)) and that its leading term at this point has sign equal to $(-1)^{[F:\mathbb{Q}]\frac{m}{2}-r_2}$ if m is even and to $(-1)^{[F:\mathbb{Q}]\frac{m-1}{2}}$ if m is odd, as per the explicit formula (2.4).

In addition, following Remark 2.8, we can (and will) assume that $\zeta_F^*(1-m)/R_m(F)$ is a rational number.

To prove Theorem 2.3 it is thus enough to show that the 2-adic component of the Bloch-Kato Conjecture for $h^0(\mathrm{Spec}(F))(1-m)$ is valid if and only if the 2-adic valuation of the rational number $\zeta_F^*(1-m)/R_m(F)$ is as implied by (2.4).

After several preliminary steps, this fact will be proved in §2.5.

2.3. **The role of Chern class maps.** Regarding F as fixed we set

$$K_{m,j} := K_{2m-j}(\mathcal{O}_F)\otimes \mathbb{Z}_2 \quad \text{and} \quad H^j_m := H^j(\mathcal{O}_F[1/2], \mathbb{Z}_2(m))$$

for each strictly positive integer m and each $j\in\{1,2\}$. We also set

$$Y_m := H^0(G_{\mathbb{C}/\mathbb{R}}, \prod_{F\to\mathbb{C}} \mathbb{Z}_2(m-1))$$

where $G_{\mathbb{C}/\mathbb{R}}$ acts diagonally on the product via its natural action on $\mathbb{Z}_2(m-1)$ and via post-composition on the set of embeddings $F\to\mathbb{C}$.

We recall that in [28] Kahn uses the Bloch-Lichtenbaum-Friedlander-Suslin-Voevodsky spectral sequences to construct for each pair of integers i and j with $j\in\{1,2\}$ and $i\ge j$ a homomorphism of $\mathbb{Z}_2$-modules $\mathrm{c}^{\mathrm{K}}_{i,j} = \mathrm{c}^{\mathrm{K}}_{F,i,j}$ of the form (2.2).

We write $R\Gamma_c(\mathcal{O}_F[1/2], \mathbb{Z}_2(1-m))$ for the compactly supported étale cohomology of $\mathbb{Z}_2(1-m)$ on $\mathrm{Spec}(\mathcal{O}_F[1/2])$ (as defined, for example, in [13, (3)]).

We recall that this complex, and hence also its (shifted) linear dual

$$C^\bullet_m := R\,\mathrm{Hom}_{\mathbb{Z}_2}(R\Gamma_c(\mathcal{O}_F[1/2], \mathbb{Z}_2(1-m)), \mathbb{Z}_2[-2])$$

belongs to the category $\mathrm{D}^{\mathrm{perf}}(\mathbb{Z}_2)$.

In the sequel we shall write $\mathrm{D}(-)$ for the Grothendieck-Knudsen-Mumford determinant functor on $\mathrm{D}^{\mathrm{perf}}(\mathbb{Z}_2)$, as constructed in [30]. (Note however that, since $\mathbb{Z}_2$ is local, one does not lose any significant information by (suppressing gradings and) regarding the values of $\mathrm{D}(-)$ as free rank one $\mathbb{Z}_2$-modules and so this is what we do.)

Proposition 2.14. *The map $\mathrm{c}^{\mathrm{K}}_{m,1}$ combines with the Artin-Verdier duality theorem to induce a canonical identification of $\mathbb{Q}_2$-spaces*

$$\mathbb{Q}_2\cdot \mathrm{D}(C^\bullet_m) = \mathbb{Q}_2\cdot\big((\bigwedge\nolimits^{d_m}_{\mathbb{Z}_2} K_{m,1})\otimes_{\mathbb{Z}_2}\mathrm{Hom}_{\mathbb{Z}_2}(\bigwedge\nolimits^{d_m}_{\mathbb{Z}_2}\cdot Y_m, \mathbb{Z}_2)\big) \tag{2.15}$$

with respect to which $\mathrm{D}(C^\bullet_m)$ is equal to

$$(2^{r_1})^{t^1_m(F)}\frac{|K_{m,2}|}{|(K_{m,1})_{\mathrm{tor}}|}\cdot(\bigwedge\nolimits^{d_m}_{\mathbb{Z}_2} K_{m,1,\mathrm{tf}})\otimes_{\mathbb{Z}_2}\mathrm{Hom}_{\mathbb{Z}_2}(\bigwedge\nolimits^{d_m}_{\mathbb{Z}_2} Y_m, \mathbb{Z}_2) \tag{2.16}$$

where the integer $t^1_m(F)$ is as defined in Theorem 2.3.

Proof. We abbreviate $\mathrm{c}^{\mathrm{K}}_{m,j}$ to c_j and set $b_m := r_1$ if m is odd and $b_m := 0$ if m is even.

Then, a straightforward computation of determinants shows that

$$\begin{aligned}
\mathrm{D}(C^\bullet_m) =& \mathrm{D}(H^0(C^\bullet_m)[0])\otimes \mathrm{D}(H^1(C^\bullet_m)[-1]) = \mathrm{D}(H^1_m[0])\otimes\mathrm{D}(H^2_m[-1])\otimes 2^{-b_m}\mathrm{D}(Y_m[-1])\\
=&(\mathrm{D}(\ker(\mathrm{c}_1)[0])^{-1}\otimes\mathrm{D}(K_{m,1}[0])\otimes\mathrm{D}(\mathrm{cok}(\mathrm{c}_1)[0]))\\
&\otimes(\mathrm{D}(\ker(\mathrm{c}_2)[-1])^{-1}\otimes\mathrm{D}(K_{m,2}[-1])\otimes\mathrm{D}(\mathrm{cok}(\mathrm{c}_2)[-1]))\otimes 2^{-b_m}\mathrm{D}(Y_m[-1])\\
=&2^{-b_m}\frac{|\ker(\mathrm{c}_1)|\cdot|\mathrm{cok}(\mathrm{c}_2)|}{|\ker(\mathrm{c}_2)|\cdot|\mathrm{cok}(\mathrm{c}_1)|}\frac{|K_{m,2}|}{|(K_{m,1})_{\mathrm{tor}}|}\cdot(\bigwedge\nolimits^{d_m}_{\mathbb{Z}_2}K_{m,1,\mathrm{tf}})\otimes_{\mathbb{Z}_2}\mathrm{Hom}_{\mathbb{Z}_2}(\bigwedge\nolimits^{d_m}_{\mathbb{Z}_2}Y_m,\mathbb{Z}_2).
\end{aligned}$$

Here the first equality is valid because $C^\bullet_m$ is acyclic outside degrees zero and one, the second follows from the descriptions of Lemma 2.19 below, the third is induced by the tautological exact sequences $0\to\ker(c_j)\to K_{m,j}\xrightarrow{c_j} H^j_m\to\mathrm{cok}(c_j)\to 0$ for $j=1,2$ and the last equality follows from the fact that for any finitely generated $\mathbb{Z}_2$-module M and integer a the lattice $\mathrm{D}(M[a])$ is equal to $(|M_{\mathrm{tor}}|)^{-1}\cdot\bigwedge^b_{\mathbb{Z}_2} M_{\mathrm{tf}}$ with $b=\dim_{\mathbb{Q}_2}(\mathbb{Q}_2\cdot M)$ if a is even and to $|M_{\mathrm{tor}}|\cdot \mathrm{Hom}_{\mathbb{Z}_2}(\bigwedge^b_{\mathbb{Z}_2} M_{\mathrm{tf}},\mathbb{Z}_2)$ if a is odd.

It thus suffices to show the product of 2^{-b_m} and $|\ker(c_1)|\cdot|\mathrm{cok}(c_2)|/(|\ker(c_2)|\cdot|\mathrm{cok}(c_1)|)$ is $(2^{r_1})^{t^1_m(F)}$ and this follows by explicit computation using the fact that [28, Th. 1] implies

$$\begin{cases} |\ker(\mathrm{c}_j)|=|\mathrm{coker}(\mathrm{c}_j)|=1, & \text{if } 2m-j\equiv 0,1,2,7 \pmod 8\\ |\ker(\mathrm{c}_j)|=2^{r_1},\ |\mathrm{coker}(\mathrm{c}_j)|=1, & \text{if } 2m-j\equiv 3 \pmod 8\\ |\ker(\mathrm{c}_j)|=2^{r_{1,4}},\ |\mathrm{coker}(\mathrm{c}_j)|=1, & \text{if } 2m-j\equiv 4 \pmod 8\\ |\ker(\mathrm{c}_j)|=1,\ |\mathrm{coker}(\mathrm{c}_j)|=2^{r_{1,5}}, & \text{if } 2m-j\equiv 5 \pmod 8\\ |\ker(\mathrm{c}_j)|=1,\ |\mathrm{coker}(\mathrm{c}_j)|=2^{r_1}, & \text{if } 2m-j\equiv 6 \pmod 8\end{cases} \tag{2.17}$$

where the integers $r_{1,4}$ and $r_{1,5}$ are zero unless $r_1>0$ in which case one has $r_{1,4}\geq 0$, $r_{1,5}>0$ and $r_{1,4}+r_{1,5}=r_1$. □

Remark 2.18. In [39] Rognes and Weibel use a slightly different approach to Kahn to construct maps of the form $\mathrm{c}^{\mathrm{K}}_{i,j}$. The results obtained in loc. cit. can be used to give an alternative proof of Proposition 2.14.

In the sequel we write Σ_∞, $\Sigma_{\mathbb{R}}$ and $\Sigma_{\mathbb{C}}$ for the sets of archimedean, real archimedean and complex archimedean places of F respectively.

Lemma 2.19. *The Artin-Verdier duality theorem induces the following identifications.*

(i) $H^0(C^\bullet_m)=H^1(\mathcal{O}_F[1/2],\mathbb{Z}_2(m))$.
(ii) $H^1(C^\bullet_m)_{\mathrm{tor}}=H^2(\mathcal{O}_F[1/2],\mathbb{Z}_2(m))$.
(iii) $H^1(C^\bullet_m)_{\mathrm{tf}}$ *is the submodule*

$$\bigoplus_{w\in\Sigma_{\mathbb{R}}} 2\cdot H^0(G_{\mathbb{C}/\mathbb{R}},\mathbb{Z}_2(m-1))\oplus\bigoplus_{w\in\Sigma_{\mathbb{C}}} H^0(G_{\mathbb{C}/\mathbb{R}},\mathbb{Z}_2(m-1)\cdot\sigma_w\oplus\mathbb{Z}_2(m-1)\cdot\overline{\sigma_w})$$

of

$$Y_m=\bigoplus_{w\in\Sigma_{\mathbb{R}}} H^0(G_{\mathbb{C}/\mathbb{R}},\mathbb{Z}_2(m-1))\oplus\bigoplus_{w\in\Sigma_{\mathbb{C}}} H^0(G_{\mathbb{C}/\mathbb{R}},\mathbb{Z}_2(m-1)\cdot\sigma_w\oplus\mathbb{Z}_2(m-1)\cdot\overline{\sigma_w}),$$

where for each place $w\in\Sigma_{\mathbb{C}}$ *we choose a corresponding embedding* $\sigma_w: F\to\mathbb{C}$ *and write* $\overline{\sigma_w}$ *for its complex conjugate.*

Proof. For each w in Σ_∞ we write $R\Gamma_{\mathrm{Tate}}(F_w,\mathbb{Z}_2(1-m))$ for the standard complete resolution of $\mathbb{Z}_2(1-m)$ that computes Tate cohomology over F_w and $R\Gamma_\Delta(F_w,\mathbb{Z}_2(1-m))$ for the mapping fibre of the natural morphism $R\Gamma(F_w,\mathbb{Z}_2(1-m))\to R\Gamma_{\mathrm{Tate}}(F_w,\mathbb{Z}_2(1-m))$.

We recall from, for example, [13, Prop. 4.1], that the Artin-Verdier Duality Theorem gives an exact triangle in $\mathrm{D}^{\mathrm{perf}}(\mathbb{Z}_2)$ of the form

$$C^\bullet_m\to\bigoplus_{v\in\Sigma_\infty} R\,\mathrm{Hom}_{\mathbb{Z}_2}(R\Gamma_\Delta(F_w,\mathbb{Z}_2(1-m)),\mathbb{Z}_2[-1])\to R\Gamma(\mathcal{O}_F[1/2],\mathbb{Z}_2(m))[2]\to C^\bullet_m[1]. \tag{2.20}$$

In addition, explicit computation shows that $R\,\mathrm{Hom}_{\mathbb{Z}_2}(R\Gamma_\Delta(F_w,\mathbb{Z}_2(1-m)),\mathbb{Z}_2[-1])$ is represented by the complex

$$\begin{cases}\mathbb{Z}_2(m-1)[-1], & \text{if } w\in\Sigma_{\mathbb{C}}\\ (\mathbb{Z}_2(m-1)\xrightarrow{\delta^0_m}\mathbb{Z}_2(m-1)\xrightarrow{\delta^1_m}\mathbb{Z}_2(m-1)\xrightarrow{\delta^0_m}\cdots)[-1], & \text{if } w\in\Sigma_{\mathbb{R}},\end{cases}$$

with $\delta^i_m := 1-(-1)^i\tau$ for $i = 0,1$ where τ denotes complex conjugation.

Given this description, and the fact that $H^2(\mathcal{O}_F[1/2], \mathbb{Z}_2(m))$ is finite, the long exact cohomology sequence of (2.20) leads directly to the identifications in claims (i) and (ii) and also gives an exact sequence of $\mathbb{Z}_2$-modules as in the upper row of the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(C^\bullet_m)_{\rm tf} & \longrightarrow & \bigoplus_{w\in\Sigma_\infty} H^0(F_w,\mathbb{Z}_2(m-1)) & \longrightarrow & H^3(\mathcal{O}_F[1/2],\mathbb{Z}_2(m)) \\ & & & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\cong} \\ & & & & \bigoplus_{w\in\Sigma_\infty} H^3(F_w,\mathbb{Z}_2(m)) & = & \bigoplus_{w\in\Sigma_\infty} H^3(F_w,\mathbb{Z}_2(m)) \end{array}$$

in which the right hand vertical isomorphism is the canonical isomorphism and θ is defined to make the square commute.

Since $H^3(F_w, \mathbb{Z}_2(m))$ is isomorphic to $\mathbb{Z}_2/2\mathbb{Z}_2$ if $w \in \Sigma_\mathbb{R}$ and m is odd and vanishes in all other cases and θ is known to respect the direct sum decompositions of its source and target (see the proof of [13, Lem. 18]), this diagram implies the description of $H^1(C^\bullet_m)_{\rm tf}$ given in claim (iii). □

Remark 2.21. The proof of Lemma 2.19 also shows that [26, Rem. following Prop. 1.2.10] requires modification. Specifically, and in terms of the notation of loc. cit., for the given statement to be true the term $\det(T_2(r)^+)$ must be replaced by $|\hat{H}^0(\mathbb{R}, T_2(r))|\cdot\det(T_2(r)^+)$ rather than by $\det(\hat{H}^0(\mathbb{R}, T_2(r)))$, as asserted at present.

2.4. **Pairings on Betti cohomology and Beilinson's regulator.** We start with an observation concerning a natural perfect pairing on Betti-cohomology.

To do this we write $\mathcal{E}_\mathbb{R}$ for the set of embeddings $F \to \mathbb{C}$ that factor through $\mathbb{R} \subset \mathbb{C}$ and set $\mathcal{E}_\mathbb{C} := \mathrm{Hom}(F, \mathbb{C}) \setminus \mathcal{E}_\mathbb{R}$. For each integer a we set $W_{a,\mathbb{R}} := \prod_{\mathcal{E}_\mathbb{R}} (2\pi i)^a\mathbb{Z}$ and $W_{a,\mathbb{C}} := \prod_{\mathcal{E}_\mathbb{C}} (2\pi i)^a\mathbb{Z}$ and endow the direct sum $W_a := W_{a,\mathbb{R}} \oplus W_{a,\mathbb{C}}$ with the diagonal action of $G_{\mathbb{C}/\mathbb{R}}$ that uses its natural action on $(2\pi i)^{m-1}\mathbb{Z}$ and post-composition on the embeddings $F \to \mathbb{C}$.

We write τ for the non-trivial element of $G_{\mathbb{C}/\mathbb{R}}$ and for each $G_{\mathbb{C}/\mathbb{R}}$-module M we use $M^\pm$ to denote the submodule comprising elements upon which τ acts as multiplication by ±1.

Then the perfect pairing

$$(\mathbb{Q} \otimes_\mathbb{Z} W_a) \times (\mathbb{Q} \otimes_\mathbb{Z} W_{1-a}) \to (2\pi i)\mathbb{Q}$$

that sends each element $((c_\sigma), (c'_\tau))$ to $\sum_\sigma c_\sigma c'_\sigma$ restricts to induce an identification

$$\mathrm{Hom}_\mathbb{Z}(W_a^+, (2\pi i)\mathbb{Z}) = W^-_{1-a,\mathbb{R}} \oplus (W_{1-a,\mathbb{C}}/(1+\tau)W_{1-a,\mathbb{C}})$$

and hence also

$$\mathrm{Hom}_\mathbb{Z}(W_a^+, \mathbb{Z}) = W^+_{-a,\mathbb{R}} \oplus (W_{-a,\mathbb{C}}/(1-\tau)W_{-a,\mathbb{C}}).$$

In particular, after identifying $\mathbb{Q}\otimes(W_{-a,\mathbb{C}}/(1-\tau)W_{-a,\mathbb{C}})$ and $\mathbb{Q}\otimes W^+_{-a,\mathbb{C}}$ in the natural way, one obtains an isomorphism

$$\mathbb{Q}\otimes W^+_{-a} \cong \mathrm{Hom}_\mathbb{Q}(\mathbb{Q}\otimes W_a^+, \mathbb{Q}) \tag{2.22}$$

that identifies W^+_{-a} with a sublattice of $\mathrm{Hom}_\mathbb{Z}(W_a^+, \mathbb{Z})$ in such a way that

$$|\mathrm{Hom}_\mathbb{Z}(W_a^+, \mathbb{Z})/W^+_{-a}| = 2^{r_2}. \tag{2.23}$$

Next, for each natural number m with $m > 1$ we consider the composite homomorphism

$$\mathrm{reg}_m : K_{2m-1}(\Gamma) \to H^1_{\mathcal{D}}(\mathrm{Spec}(\Gamma), (2\pi i)\mathbb{R}) = \Gamma/(2\pi i)^m\mathbb{R} \to (2\pi i)^{m-1}\mathbb{R}$$

where the first arrow is the Beilinson regulator map and the second is the isomorphism induced by the decomposition $\Gamma = (2\pi i)^m\mathbb{R} \oplus (2\pi i)^{m-1}\mathbb{R}$. We then write

$$\beta_m : \mathbb{R} \otimes K_{2m-1}(\mathcal{O}_F) \to \mathbb{R} \otimes W^+_{m-1} \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R} \otimes W^+_{1-m}, \mathbb{R}) \tag{2.24}$$

for the composite homomorphism of $\mathbb{R}[G]$-modules where the first map is induced by the composites of reg_m with the maps $K_{2m-1}(F) \to K_{2m-1}(\mathbb{C})$ that are induced by each embedding $F \to \mathbb{C}$ and the second by the isomorphism (2.22) with $a = 1 - m$.

The map β_m is bijective and we write $\beta_{m,*}$ for the induced isomorphism of $\mathbb{R}$-spaces

$$\mathbb{R} \otimes (\textstyle\bigwedge^{d_m}_{\mathbb{Z}} K_{2m-1}(\mathcal{O}_F)) \otimes \mathrm{Hom}_{\mathbb{Z}}(\bigwedge^{d_m}_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(W^+_{1-m}, \mathbb{Z}), \mathbb{Z}) \to \mathbb{R}$$

that is induced by β_m.

2.5. **Completion of the proof of Theorem 2.3.** After making explicit the formulation of [14, Conj. 1] (which originates with Fontaine and Perrin-Riou [22, Prop. III.3.2.5]) and the construction of [14, Lem. 18], one finds that the Bloch-Kato Conjecture for $h^0(\mathrm{Spec}(F))(1-m)$ uses the composite map β_m in (2.24) rather than simply the first map that occurs in its definition.

In addition, if one fixes a topological generator η of $\mathbb{Z}_2(m-1)$, then by mapping η to the element of $\mathrm{Hom}_{\mathbb{Q}}((2\pi i)^{1-m}\mathbb{Q}, \mathbb{Q})$ that sends $(2\pi i)^{1-m}$ to 1 one obtains an identification of Y_m with $\mathbb{Z}_2 \otimes \mathrm{Hom}_{\mathbb{Z}}(W^+_{1-m}, \mathbb{Z})$.

Given these observations, the discussion of [26, §1.2] shows the Bloch-Kato Conjecture asserts that if one fixes an identification of $\mathbb{C}$ with $\mathbb{C}_2$, then $\zeta^*_F(1-m)$ is a generator over $\mathbb{Z}_2$ of the image of $\mathrm{D}(C^\bullet_m)$ under the composite isomorphism

$$\mathbb{C}_2 \cdot \mathrm{D}(C^\bullet_m) \cong \mathbb{C}_2 \cdot ((\textstyle\bigwedge^{d_m}_{\mathbb{Z}_2} K_{m,1}) \otimes_{\mathbb{Z}_2} \mathrm{Hom}_{\mathbb{Z}_2}(\bigwedge^{d_m}_{\mathbb{Z}_2} \cdot Y_m, \mathbb{Z}_2)) \cong \mathbb{C}_2.$$

Here the first map differs from the scalar extension of (2.15) only in that, in its construction, the map $\mathrm{c}^{\mathrm{S}}_{F,m,1}$ is used in place of $\mathrm{c}^{\mathrm{K}}_{F,m,1}$ and the second isomorphism is $\mathbb{C}_2 \otimes_{\mathbb{R}} \beta_{m,*}$.

In the sequel we write $\det_{\mathbb{Q}_2}(\alpha)$ for the determinant of an automorphism of a finite dimensional $\mathbb{Q}_2$-vector space.

Then, by combining the above observations together with the result of Proposition 2.14 one finds that the Bloch-Kato Conjecture predicts the image under $\mathbb{C}_2 \otimes_{\mathbb{R}} \beta_{m,*}$ of the lattice (2.16) to be generated over $\mathbb{Z}_2$ by the product $\zeta^*_F(1-m) \cdot \det_{\mathbb{Q}_2}((\mathbb{Q}_2 \cdot \mathrm{c}^{\mathrm{S}}_{F,m,1}) \circ (\mathbb{Q}_2 \cdot \mathrm{c}^{\mathrm{K}}_{F,m,1})^{-1})^{-1}$.

In addition, since the regulator $R_m(F)$ is defined with respect to the lattice W^+_{m-1} rather than $\mathrm{Hom}_{\mathbb{Z}}(W^+_{1-m}, \mathbb{Z})$, the index formula (2.23) implies that

$$(\mathbb{C}_2 \otimes_{\mathbb{R}} \beta_{m,*})((\textstyle\bigwedge^{d_m}_{\mathbb{Z}_2} K_{m,1,\mathrm{tf}}) \otimes_{\mathbb{Z}_2} \mathrm{Hom}_{\mathbb{Z}_2}(\bigwedge^{d_m}_{\mathbb{Z}_2} Y_m, \mathbb{Z}_2)) = 2^{r_2} \cdot R_m(F) \cdot \mathbb{Z}_2 \subset \mathbb{C}_2.$$

To deduce the explicit formula (2.4) from these facts (by direct substitution) we now need only note that Lemma 2.25 below implies

$$2^{a_m(F)} |\mathrm{cok}(\mathrm{c}^{\mathrm{S}}_{F,m,1,\mathrm{tf}})|^{-1} \cdot \det_{\mathbb{Q}_2}((\mathbb{Q}_2 \cdot \mathrm{c}^{\mathrm{S}}_{F,m,1}) \circ (\mathbb{Q}_2 \cdot \mathrm{c}^{\mathrm{K}}_{F,m,1})^{-1}) \in \mathbb{Z}_2^\times.$$

Finally we note that the congruence which occurs in (2.5) is a direct consequence of Lemma 2.25(ii) below and this then completes the proof of Theorem 2.3.

Lemma 2.25.

(i) One has $\det_{\mathbb{Q}_2}((\mathbb{Q}_2\cdot \mathrm{c}^{\mathrm{DF}}_{F,m,1})\circ(\mathbb{Q}_2\cdot \mathrm{c}^{\mathrm{K}}_{F,m,1})^{-1})\in\mathbb{Z}_2^\times$.

(ii) Define a non-negative integer $a_m(F)$ *by the equality* $2^{a_m(F)} = |\mathrm{cok}(\mathrm{c}^{\mathrm{K}}_{F,m,1,\mathrm{tf}})|$. *Then*

$$2^{a_m(F)}|\mathrm{cok}(\mathrm{c}^{\mathrm{S}}_{F,m,1,\mathrm{tf}})|^{-1}\cdot\det_{\mathbb{Q}_2}((\mathbb{Q}_2\cdot \mathrm{c}^{\mathrm{S}}_{F,m,1})\circ(\mathbb{Q}_2\cdot \mathrm{c}^{\mathrm{DF}}_{F,m,1})^{-1})\in\mathbb{Z}_2^\times$$

and $a_m(F)$ *is equal to* 0 *except possibly when both* $m\equiv 3$ *(mod 4) and* $r_1>0$ *in which case one has* $a_m(F)=r_{1,5}-1\geq 0$.

Proof. Set $F' := F(\sqrt{-1})$ and $\Delta := G_{F'/F}$. With $\theta_{F'}$ denoting either $\mathrm{c}^{\mathrm{S}}_{F',i,1}$, $\mathrm{c}^{\mathrm{DF}}_{F',i,1}$ or $\mathrm{c}^{\mathrm{K}}_{F',i,1}$ we write θ_F for the corresponding map defined at the level of F. Then there is a commutative diagram

$$\begin{array}{ccc} K_{2m-1}(\mathcal{O}_{F'})\otimes\mathbb{Z}_2 & \xrightarrow{\theta_{F'}} & H^1(\mathcal{O}_{F'}[1/2],\mathbb{Z}_2(m)) \\ \uparrow & & \uparrow \\ K_{2m-1}(\mathcal{O}_{F})\otimes\mathbb{Z}_2 & \xrightarrow{\theta_{F}} & H^1(\mathcal{O}_{F}[1/2],\mathbb{Z}_2(m)) \end{array}$$

in which the left hand vertical arrow is induced by the inclusion $\mathcal{O}_F\subseteq\mathcal{O}_{F'}$ and the right hand vertical arrow is the natural restriction map. In addition, since $\theta_{F'}$ is natural with respect to automorphisms of F' it is Δ-equivariant and hence, upon taking Δ-fixed points, and noting that the right hand vertical arrow identifies $H^1(\mathcal{O}_F[1/2],\mathbb{Q}_2(m))$ with $H^0(\Delta,H^1(\mathcal{O}_{F'}[1/2],\mathbb{Q}_2(m)))$, the diagram identifies $\mathbb{Q}_2\cdot H^0(\Delta,\theta_{F'})$ with $\mathbb{Q}_2\cdot\theta_F$.

Now, in [19, Th. 8.7 and Rem. 8.8] Dwyer and Friedlander show that the map $\mathrm{c}^{\mathrm{DF}}_{F',m,1}$ is surjective and hence, since it maps between finitely generated $\mathbb{Z}_2$-modules of the same rank, the induced map $\mathrm{c}^{\mathrm{DF}}_{F',m,1,\mathrm{tf}}$ is bijective. In addition, since $r_1(F')=0$, the description (2.17) implies $\mathrm{c}^{\mathrm{K}}_{F',m,1}$, and hence also $\mathrm{c}^{\mathrm{K}}_{F',m,1,\mathrm{tf}}$, is bijective. One thus has $\mathrm{c}^{\mathrm{DF}}_{F',m,1,\mathrm{tf}}=\varphi\circ\mathrm{c}^{\mathrm{K}}_{F',m,1,\mathrm{tf}}$ for an automorphism φ of the $\mathbb{Z}_2[\Delta]$-lattice $\Xi_{F'}:=H^j(\mathcal{O}_{F'}[1/2],\mathbb{Z}_2(m))_{\mathrm{tf}}$. Restricting to Δ-fixed points this implies that

$$\begin{aligned}\det_{\mathbb{Q}_2}((\mathbb{Q}_2\cdot \mathrm{c}^{\mathrm{DF}}_{F,m,1})\circ(\mathbb{Q}_2\cdot \mathrm{c}^{\mathrm{K}}_{F,m,1})^{-1}) =&\det_{\mathbb{Q}_2}(H^0(\Delta,\mathbb{Q}_2\cdot \mathrm{c}^{\mathrm{DF}}_{F',m,1})\circ H^0(\Delta,\mathbb{Q}_2\cdot \mathrm{c}^{\mathrm{K}}_{F',m,1})^{-1})\\ =&\det_{\mathbb{Q}_2}(H^0(\Delta,\mathbb{Q}_2\cdot\varphi))\end{aligned}$$

and this belongs to $\mathbb{Z}_2^\times$ since $H^0(\Delta,\varphi)$ is an automorphism of the $\mathbb{Z}_2$-lattice $H^0(\Delta,\Xi_{F'})$. This proves claim (i).

In view of claim (i) it suffices to prove claim (ii) with $\mathrm{c}^{\mathrm{DF}}_{F,m,1}$ replaced by $\mathrm{c}^{\mathrm{K}}_{F,m,1}$. By (2.17) one also knows $\mathrm{c}^{\mathrm{K}}_{F,m,1}$, and hence also $\mathrm{c}^{\mathrm{K}}_{F,m,1,\mathrm{tf}}$, is surjective except possibly if $m\equiv 3$ (mod 4) and $r_1>0$. In the latter case, the $\mathbb{Z}_2$-module $K_{m,1}$ is torsion-free (by [39, Th. 0.6]) and, since $r_1>0$ and m is odd, it is straightforward to check that $|H^1_{m,\mathrm{tor}}|=2$ (see, for example, [39, Props. 1.8 and 1.9(b)]). In this case therefore, the computation (2.17) implies that $|\mathrm{cok}(\mathrm{c}^{\mathrm{K}}_{F,m,1,\mathrm{tf}})|=2^{-1}\cdot|\mathrm{cok}(\mathrm{c}^{\mathrm{K}}_{F,m,1})|=2^{r_{1,5}-1}$.

Given this, and the explicit definition of $a_m(F)$, claim (ii) follows directly from the fact that, when computed with respect to any fixed $\mathbb{Z}_2$-bases of $(K_{2m-1}(\mathcal{O}_F)\otimes\mathbb{Z}_2)_{\mathrm{tf}}$ and Ξ_F, the determinants of $\mathrm{c}^{\mathrm{K}}_{F,m,1,\mathrm{tf}}$ and $\mathrm{c}^{\mathrm{S}}_{F,m,1,\mathrm{tf}}$ are generators of $2^{a_m(F)}\cdot\mathbb{Z}_2$ and $|\mathrm{cok}(\mathrm{c}^{\mathrm{S}}_{F,m,1,\mathrm{tf}})|\cdot\mathbb{Z}_2$ respectively. □

Remark 2.26. In [28, just after Th. 1] Kahn explicitly asks whether if $m\equiv 3$ (mod 4) and $r_1>0$ the integer $r_{1,5}$ in (2.17) is always equal to 1 and points out that this amounts to asking whether, in this case, the image of $H^1(\mathcal{O}_F[1/2],\mathbb{Z}_2(m))$ in $H^1(\mathcal{O}_F[1/2],\mathbb{Z}/2) \subset F^\times/(F^\times)^2$ is contained in the subgroup generated by the classes of ± 1 and the set of totally positive elements of $F^\times$? If the answer to this question is affirmative, then in all cases one has $a_m(F) = 0$.

2.6. **An upper bound for $t_m^2(F)$.** We henceforth fix a natural number m with $m > 1$ and for each number field E abbreviate Soulé's 2-adic Chern class map

$$\mathrm{c}^{\mathrm{S}}_{E,m,1,2} : K_{2m-1}(\mathcal{O}_E) \otimes \mathbb{Z}_2 \to H^1(\mathcal{O}_E[1/2],\mathbb{Z}_2(m))$$

to c_E. We also set $E' := E(\sqrt{-1})$.

The computation made in §2.3 relied crucially on the fact that the cokernel of c_F is finite. However, whilst this fact is well-known to experts, we were not able to locate a proof in the literature (and also see Remark 2.30 below).

In addition, and as already discussed in Remark 2.7, for the purposes of numerical computations, one must not only know that $\mathrm{cok}(\mathrm{c}_F)$ is finite but also have a computable upper bound for $\mathrm{cok}(\mathrm{c}_{F,\mathrm{tf}}) = 2^{t_m^2(F)}$.

In this section we address these issues by proving the following result.

Theorem 2.27. $\mathrm{cok}(\mathrm{c}_{F,\mathrm{tf}})$ *is finite and its cardinality divides*

$$[F' : F]^{d_m(F)}((m-1)!\,)^{d_m(F')}|K_{2m-2}(\mathcal{O}_{F'})|.$$

As preparation for the proof of Theorem 2.27 we first consider universal norm subgroups in étale cohomology. To do this we write F'_∞ for the cyclotomic $\mathbb{Z}_2$-extension of F' and F_n for each non-negative integer n for the unique subfield of F'_∞ of degree 2^n over F'. We also set $\Gamma := G_{F'_\infty/F'}$ and write Λ for the Iwasawa algebra $\mathbb{Z}_2[[\Gamma]]$. For each Λ-module N and integer a we write $N(a)$ for the Λ-module $N \otimes_{\mathbb{Z}_2} \mathbb{Z}_2(a)$ upon which Γ acts diagonally.

For each finite extension E of F' we abbreviate $\mathcal{O}_E[1/2]$ to $\mathcal{O}'_E$. We then define the 'universal norm' subgroup $H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m))$ of $H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))$ to be the image of the natural projection map $\varprojlim_n H^1(\mathcal{O}'_{F'_n},\mathbb{Z}_2(m)) \to H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))$ where the limit is taken with respect to the natural corestriction maps.

Proposition 2.28. *The index of* $H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m))$ *in* $H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))$ *divides* $|K_{2m-2}(\mathcal{O}_{F'})|$.

Proof. We write $C^\bullet_\infty$ for the object of the derived category of perfect complexes of Λ-modules that is obtained as the inverse limit of the complexes $R\Gamma(\mathcal{O}'_{F'_n},\mathbb{Z}_2(m))$ with respect to the natural projection morphisms

$$R\Gamma(\mathcal{O}'_{F'_{n+1}},\mathbb{Z}_2(m)) \to \mathbb{Z}_p[\Gamma_n] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[\Gamma_{n+1}]} R\Gamma(\mathcal{O}'_{F'_{n+1}},\mathbb{Z}_2(m)) \cong R\Gamma(\mathcal{O}'_{F'_n},\mathbb{Z}_2(m)).$$

We recall that there is a natural isomorphism $\mathbb{Z}_2 \otimes^{\mathbb{L}}_\Lambda C^\bullet_\infty \cong R\Gamma(\mathcal{O}'_{F'},\mathbb{Z}_2(m))$ in $D^{\mathrm{perf}}(\mathbb{Z}_2)$ and that this induces a natural short exact sequence of $\mathbb{Z}_2$-modules

$$0 \to \mathbb{Z}_2 \otimes_{\mathbb{Z}_2[[\Gamma]]} H^1(C^\bullet_\infty) \xrightarrow{\pi} H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m)) \to H^0(\Gamma, H^2(C^\bullet_\infty)) \to 0. \tag{2.29}$$

In addition, in each degree i one has $H^i(C^\bullet_\infty) = \varprojlim_n H^i(\mathcal{O}'_{F'_n},\mathbb{Z}_2(m))$, where the limits are taken with respect to the natural corestriction maps, and so $\mathrm{im}(\pi) = H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m))$ and the Λ-module $H^2(C^\bullet_\infty)$ is isomorphic to $(\varprojlim_n H^i(\mathcal{O}'_{F'_n},\mathbb{Z}_2(1)))(m-1)$.

Now for each n class field theory identifies $H^2(\mathcal{O}'_{F'_n},\mathbb{Z}_2(1))_{\mathrm{tor}}$ with the ideal class group $\mathrm{Pic}(\mathcal{O}'_{F'_n})$ of $\mathcal{O}'_{F'_n}$ and $H^2(\mathcal{O}'_{F'_n},\mathbb{Z}_2(1))$ with a submodule of the free $\mathbb{Z}_2$-module on the set of places of F'_n that are either archimedean or 2-adic and hence, upon passing to the limit over n and then taking Γ-invariants, gives rise to an exact sequence of $\mathbb{Z}_2$-modules

$$0 \to H^0(\Gamma, X'_\infty(m-1)) \to H^0(\Gamma, H^2(C^\bullet_\infty)) \to \bigoplus_{v\in\Sigma_2\cup\Sigma_\infty(F')} H^0(\Gamma, \mathbb{Z}_2[[\Gamma/\Gamma_v]](m-1))$$

where X'_∞ is the Galois group of the maximal unramified pro-2 extension of F_∞ in which all 2-adic places split completely, Σ_2 the set of 2-adic places of F' and Γ_v the decomposition subgroup of v in Γ. Since $m > 1$ it is also clear that each module $H^0(\Gamma, \mathbb{Z}_2[[\Gamma/\Gamma_v]](m-1))$ vanishes and hence that the sequence implies $H^0(\Gamma, X'_\infty(m-1)) = H^0(\Gamma, H^2(C^\bullet_\infty))$.

In view of the exact sequence (2.29) we are thus reduced to showing that $H^0(\Gamma, X'_\infty(m-1))$ is finite and of order dividing $|K_{2m-2}(\mathcal{O}_{F'})|$.

Next we recall that, by a standard 'Herbrand Quotient' argument in Iwasawa theory (see, for example, [46, Exer. 13.12]), if a finitely generated Λ-module N is such that $H_0(\Gamma, N)$ is finite, then $H^0(\Gamma, N)$ is both finite and of order at most $|H_0(\Gamma, N)|$. In addition, since F' is totally imaginary, the argument of Schneider in [40, §6, Lem. 1] (see also the discussion of Le Floc'h, Movahhedi and Nguyen Quang Do in [34, just before Lem. 1.2]) shows that $H_0(\Gamma, X'_\infty(m-1))$ is naturally isomorphic to the 'étale wild kernel'

$$\mathrm{WK}^{\text{ét}}_{2m-2}(F') := \ker(H^2(\mathcal{O}'_{F'}, \mathbb{Z}_2(m)) \to \bigoplus_{v\in\Sigma_2(F')\cup\Sigma_\infty(F')} H^2(F'_w, \mathbb{Z}_2(m)))$$

of F', where the arrow denotes the natural diagonal localisation map.

Hence, to deduce the claimed result, we need only recall that, as F' is totally imaginary, the group $H^2(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$ is naturally isomorphic to the finite group $K_{2m-2}(\mathcal{O}_{F'}) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ (as a consequence of (2.17)). □

Turning now to the proof of Theorem 2.27, we consider for each n the following diagram

$$\begin{array}{ccccc}
K_{2m-1}(\mathcal{O}'_{F'_n}, \mathbb{Z}/2^n) & \xrightarrow{m\cdot \mathrm{c}_{F'_n,2^n}} & H^1(\mathcal{O}'_{F'_n}, (\mathbb{Z}/2^n)(m)) & \xrightarrow{\iota_n} & H^1(F'_n, (\mathbb{Z}/2^n)(m)) \\
\downarrow & & \downarrow & & \downarrow \\
K_{2m-1}(\mathcal{O}'_{F'}, \mathbb{Z}/2^n) & \xrightarrow{m\cdot \mathrm{c}_{F',2^n}} & H^1(\mathcal{O}'_{F'}, (\mathbb{Z}/2^n)(m)) & \xrightarrow{\iota} & H^1(F', (\mathbb{Z}/2^n)(m))\,.
\end{array}$$

Here we write $\mathrm{c}_{E,2^n}$ for the Chern class maps $K_{2m-1}(\mathcal{O}'_E, \mathbb{Z}/2^n) \to H^1(\mathcal{O}'_E, (\mathbb{Z}/2^n)(m))$ of Soulé, as discussed by Weibel in [49], the arrows ι_n and ι are the natural inflation maps, the left hand vertical arrow is the natural transfer map and the remaining vertical arrows are the natural corestriction maps. In particular, the results of [49, Prop. 2.1.1 and 4.4] combine to imply that the outer rectangle of this diagram commutes, and hence since the maps ι_n and ι are injective, that the first square also commutes.

Since the first square is compatible with change of n in the natural way we may then pass to the inverse limit over n to obtain a commutative diagram

$$\begin{array}{ccc}
\varprojlim_n K_{2m-1}(\mathcal{O}'_{F'_n}, \mathbb{Z}/2^n) & \xrightarrow{(m\cdot \mathrm{c}_{F'_n,2^n})_n} & \varprojlim_n H^1(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(m)) \\
\downarrow & & \downarrow \\
K_{2m-1}(\mathcal{O}'_{F'}) \otimes_{\mathbb{Z}} \mathbb{Z}_2 & \xrightarrow{m\cdot \mathrm{c}_{F'}} & H^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))\,.
\end{array}$$

Here we use the fact that, since $K_{2m-1}(\mathcal{O}'_{F'})$ is finitely generated, $\varprojlim_n K_{2m-1}(\mathcal{O}'_{F'},\mathbb{Z}/2^n)$ identifies with $K_{2m-1}(\mathcal{O}'_{F'})\otimes_{\mathbb{Z}}\mathbb{Z}_2$ in such a way that the limit $(m\cdot \mathrm{c}_{F',2^n})_n$ is equal to $m\cdot \mathrm{c}_{F'}$.

Now the image of the right hand vertical arrow in this diagram is $H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m))$ and, as each F'_n contains all roots of unity of order 2^n, from [49, Cor. 5.6] one knows that the exponent of $\mathrm{cok}((m\cdot \mathrm{c}_{F'_n,2^n})_n)$ divides $m!$. From the commutativity of the above diagram we can therefore deduce $\mathrm{im}(m\cdot \mathrm{c}_{F'})$ contains $m!\cdot H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m))$ and hence also that $\mathrm{im}(\mathrm{c}_{F',\mathrm{tf}})$ contains $(m-1)!\cdot H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}}$. This inclusion implies that $|\mathrm{cok}(\mathrm{c}_{F',\mathrm{tf}})|$ divides

$$|(H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}}/H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}})|\cdot|(H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}}/(m-1)!\cdot H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}})|$$

and hence also $|(H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))/H^1_\infty(\mathcal{O}'_{F'},\mathbb{Z}_2(m)))|\cdot((m-1)!\,)^{d_m(F')}$.

Recalling Proposition 2.28 it now follows $|\mathrm{cok}(\mathrm{c}_{F',\mathrm{tf}})|$ divides $((m-1)!\,)^{d_m(F')}|K_{2m-2}(\mathcal{O}_{F'})|$, as claimed by Theorem 2.27 in the case $F'=F$.

To deduce the general case of Theorem 2.27 we assume $F'\neq F$, write τ for the unique non-trivial element of Δ and note [49, Prop. 4.4] implies there is a commutative diagram

$$\begin{array}{ccc} K_{2m-1}(\mathcal{O}'_{F'})\otimes_{\mathbb{Z}}\mathbb{Z}_2 & \xrightarrow{\mathrm{c}_{F',\mathrm{tf}}} & H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}} \\ \downarrow{\scriptstyle T^1_\Delta} & & \downarrow{\scriptstyle T^2_\Delta} \\ K_{2m-1}(\mathcal{O}'_{F})\otimes_{\mathbb{Z}}\mathbb{Z}_2 & \xrightarrow{\mathrm{c}_{F,\mathrm{tf}}} & H^1(\mathcal{O}'_{F},\mathbb{Z}_2(m))_{\mathrm{tf}} \end{array}$$

where the maps T^i_Δ are induced by the respective actions of $1+\tau\in\mathbb{Z}_2[\Delta]$. The commutativity of this diagram implies the index of $\mathrm{c}_{F,\mathrm{tf}}(\mathrm{im}(T^1_\Delta))$ in $\mathrm{im}(T^2_\Delta)$ divides $|\mathrm{cok}(\mathrm{c}_{F',\mathrm{tf}})|$. Thus, since $H^1(\mathcal{O}'_F,\mathbb{Z}_2(m))_{\mathrm{tf}}$ identifies with a (finite index) subgroup of $H^0(\Delta,H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}})$, to deduce Theorem 2.27 from the special case $F=F'$ it is enough to show that the Tate cohomology group

$$\hat{H}^0(\Delta,H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}}) := H^0(\Delta,H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}})/\mathrm{im}(T^2_\Delta)$$

has cardinality dividing $2^{d_m(F)}$ and this is true because this quotient has exponent dividing $|\Delta|=2$ and the $\mathbb{Z}_2$-lattice $H^0(\Delta,H^1(\mathcal{O}'_{F'},\mathbb{Z}_2(m))_{\mathrm{tf}})$ has rank $d_m(F)$.

This completes the proof of Theorem 2.27.

Remark 2.30. The argument of Huber and Wildeshaus in [27, Th. B.4.8 and Lem. B.4.7] aims to show, amongst other things, that $\mathrm{cok}(\mathrm{c}^{\mathrm{S}}_{F,m,1,2})$ is finite. However, this argument uses in a key way results of Dwyer and Friedlander from [19, Th. 8.7 and Rem. 8.8] that relate to $\mathrm{c}^{\mathrm{DF}}_{F,m,1,2}$ rather than $\mathrm{c}^{\mathrm{S}}_{F,m,1,2}$. To complete this argument one would thus need to investigate the relation between $\mathrm{cok}(\mathrm{c}^{\mathrm{DF}}_{F,m,1,2})$ and $\mathrm{cok}(\mathrm{c}^{\mathrm{S}}_{F,m,1,2})$.

3. K-theory, wedge complexes, and configurations of points

Let $\mathcal{F}$ be an infinite field. Then it is well-known by work of Bloch and (subsequently) Suslin that $K_3(\mathcal{F})$ is closely related to the Bloch group $B(\mathcal{F})$ (as defined in §3.3.4 below). However, the group $B(\mathcal{F})$ often contains non-trivial elements of finite order and so can be difficult for the purposes of explicit computation. With this in mind, in this section we shall introduce a slight variant $\overline{B}(\mathcal{F})$ of $B(\mathcal{F})$ over which we have better control.

We shall also construct a canonical homomorphism $\psi_{\mathcal{F}}$ from $\overline{B}(\mathcal{F})$ to $K_3(\mathcal{F})^{\rm ind}_{\rm tf}$ (see Theorem 3.25) and are motivated to conjecture, on the basis of extensive computational evidence, that $\psi_{\mathcal{F}}$ is bijective if $\mathcal{F}$ is a number field (see Conjecture 3.34). We note, in particular, that these observations provide the first concrete evidence to suggest both that the groups defined by Suslin in terms of group homology and by Bloch in terms of relative K-theory should be related in a very natural way, and also that Bloch's group should account for all of $K_3^{\rm ind}(F)$, at least modulo torsion (cf. Remark 3.35).

If $\mathcal{F}$ is imaginary quadratic, then the groups $\overline{B}(\mathcal{F})$ and $K_3(\mathcal{F})^{\rm ind}_{\rm tf}$ are both isomorphic to $\mathbb{Z}$ (see Corollary 3.30 for $\overline{B}(\mathcal{F})$) and we shall later use $\psi_{\mathcal{F}}$ to reduce the problem of finding a generating element of $K_3(\mathcal{F})^{\rm ind}_{\rm tf}$ to computational issues in $\overline{B}(\mathcal{F})$.

3.1. **Towards explicit versions of $K_3(\mathcal{F})$ and reg_2.** In this section we review a result of the first three authors in [16].

3.1.1. We first recall some basic facts concerning the K_3-group of a general field $\mathcal{F}$. To this end we write $K_3(\mathcal{F})^{\rm ind}$ for the quotient of $K_3(\mathcal{F})$ by the image of the Milnor K-group $K_3^M(\mathcal{F})$ of $\mathcal{F}$.

We recall that if the field is a number field F, then the abelian group $K_3^M(F)$ has exponent 1 or 2 and order $2^{r_1(F)}$ (cf. [47, p.146]), so that $K_3(F)^{\rm ind}_{\rm tf}$ identifies with $K_3(F)_{\rm tf}$, hence is a free abelian group of rank $r_2(F)$ as a consequence of Theorem 2.1(ii).

We further recall that for any field $\mathcal{F}$ the torsion subgroup of $K_3(\mathcal{F})^{\rm ind}$ is explicitly described by Levine in [35, Cor. 4.6].

Example 3.1. If $\mathcal{F}$ is equal to an imaginary quadratic extension k of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$, then the latter result gives isomorphisms

$$K_3(k)^{\rm ind}_{\rm tor} \simeq H^0(\mathrm{Gal}(\overline{\mathbb{Q}}/k), \mathbb{Q}(2)/\mathbb{Z}(2)) = H^0(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}(2)/\mathbb{Z}(2)) = \mathbb{Z}/24\mathbb{Z}$$

where the first equality is valid because complex conjugation acts trivially on $\mathbb{Q}(2)/\mathbb{Z}(2)$ and the second follows by explicit computation. For any such field k the abelian group $K_3(k)^{\rm ind} = K_3(k)$ is therefore isomorphic to a direct product of the form $\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z}$.

3.1.2. For an arbitrary field $\mathcal{F}$ we set

$$\tilde{\wedge}^2 \mathcal{F}^* := \frac{\mathcal{F}^* \otimes_{\mathbb{Z}} \mathcal{F}^*}{\langle (-x) \otimes x \text{ with } x \text{ in } \mathcal{F}^* \rangle},$$

which is a quotient of the usual exterior power $\mathcal{F}^* \otimes_{\mathbb{Z}} \mathcal{F}^* / \langle x \otimes y + y \otimes x \text{ with } x, u \text{ in } \mathcal{F}^* \rangle$.

For each a and b in $\mathcal{F}^*$ we write $a \tilde{\wedge} b$ for the class of $a \otimes b$ in $\tilde{\wedge}^2 \mathcal{F}^*$. Then it is easily verified that the sum $a \tilde{\wedge} b + b \tilde{\wedge} a$ is trivial.

We next set $\mathcal{F}^\flat := \mathcal{F} \setminus \{0, 1\}$ and write $\mathbb{Z}[\mathcal{F}^\flat]$ for the free abelian group on $\mathcal{F}^\flat$. We consider the homomorphism

$$\delta_{2,\mathcal{F}} \colon \mathbb{Z}[\mathcal{F}^\flat] \to \tilde{\wedge}^2 \mathcal{F}^* \tag{3.2}$$

that for each x in $\mathcal{F}^\flat$ sends $[x]$ to $(1-x) \tilde{\wedge}\, x$.

We write $D(z) \colon \mathbb{C}^\flat \to \mathbb{R}$ for the Bloch-Wigner dilogarithm. This function is defined by Bloch in [5] by integrating $\log|w| \cdot d\arg(1-w) - \log|1-w| \cdot d\arg(w)$ along any path from a point z_0

in $\mathbb{R}^\flat$ to z. We recall that, by differentiating, it is easily shown that this function satisfies the identities

$$\begin{gathered} D(z)+D(z^{-1})=0, \qquad D(z)+D(1-z)=0, \qquad D(z)+D(\overline{z})=0, \\ D(x)-D(y)+D\Big(\frac{y}{x}\Big)-D\Big(\frac{1-y}{1-x}\Big)+D\Big(\frac{1-y^{-1}}{1-x^{-1}}\Big)=0, \end{gathered} \tag{3.3}$$

for x, y, and z in $\mathbb{C}^\flat$ with $x \neq y$.

We note, in particular, that the third identity here implies that the map iD from $\mathbb{C}^\flat$ to $\mathbb{R}(1)$ is equivariant with respect to the natural action of complex conjugation.

The following result describes an important connection between these constructions.

Theorem 3.4 ([16, Th. 4.1]). *With the above notation, the following claims are valid.*

(i) There exists a homomorphism

$$\varphi_{\mathcal{F}}\colon \ker(\delta_{2,\mathcal{F}}) \to K_3(\mathcal{F})^{\mathrm{ind}}_{\mathrm{tf}}\,,$$

that is natural up to sign, and, after fixing a choice of sign, functorial in $\mathcal{F}$.

(ii) If $\mathcal{F}$ is a number field F, then the cokernel of φ_F is finite.

(iii) There exists a universal choice of sign such that if F is any number field, and $\sigma\colon F \to \mathbb{C}$ is any embedding, then the composition

$$\mathrm{reg}_\sigma\colon \ker(\delta_{2,F}) \xrightarrow{\varphi_F} K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} = K_3(F)_{\mathrm{tf}} \xrightarrow{\sigma_*} K_3(\mathbb{C})_{\mathrm{tf}} \xrightarrow{\mathrm{reg}_2} \mathbb{R}(1)$$

is induced by sending each element $[x]$ for x in $F^\flat$ to $iD(\sigma(x))$.

3.2. **Analysing our wedge product.** In this section we obtain explicit information on the structure of $\tilde{\wedge}^2\mathcal{F}^*$ for a general field $\mathcal{F}$. With an eye towards implementation for the purposes of numerical calculations, we pay special attention to the case that $\mathcal{F}$ is a number field.

3.2.1. We first consider the abstract structure of $\tilde{\wedge}^2\mathcal{F}^*$.

Proposition 3.5. *For a field $\mathcal{F}$, we have a filtration $\{0\} = \mathrm{Fil}_0 \subseteq \mathrm{Fil}_1 \subseteq \mathrm{Fil}_2 \subseteq \mathrm{Fil}_3 = \tilde{\wedge}^2\mathcal{F}^*$, with Fil_1 the image of $\mathcal{F}^*_{\mathrm{tor}} \otimes \mathcal{F}^*_{\mathrm{tor}}$ and Fil_2 the image of $\mathcal{F}^*_{\mathrm{tor}} \otimes \mathcal{F}^*$. Then*

$$\mathrm{Fil}_2 = \frac{\mathcal{F}^*_{\mathrm{tor}} \otimes \mathcal{F}^*}{\langle (-x)\otimes x \text{ with } x \text{ in } \mathcal{F}^*_{\mathrm{tor}}\rangle}$$

and there are natural isomorphisms

$$\begin{aligned} \mathrm{Fil}_1/\mathrm{Fil}_0 &= \tilde{\wedge}^2\mathcal{F}^*_{\mathrm{tor}} = \frac{\mathcal{F}^*_{\mathrm{tor}} \otimes_{\mathbb{Z}} \mathcal{F}^*_{\mathrm{tor}}}{\langle (-x)\otimes x \text{ with } x \text{ in } \mathcal{F}^*_{\mathrm{tor}}\rangle} \\ \mathrm{Fil}_2/\mathrm{Fil}_1 &\simeq \mathcal{F}^*_{\mathrm{tor}} \otimes \mathcal{F}^*_{\mathrm{tf}} \\ \mathrm{Fil}_3/\mathrm{Fil}_2 &\simeq \frac{\mathcal{F}^*_{\mathrm{tf}} \otimes_{\mathbb{Z}} \mathcal{F}^*_{\mathrm{tf}}}{\langle x\otimes x \text{ with } x \text{ in } \mathcal{F}^*_{\mathrm{tf}}\rangle}\,, \end{aligned}$$

with the last two induced by the quotient maps $\mathcal{F}^*_{\mathrm{tor}} \otimes \mathcal{F}^* \to \mathcal{F}^*_{\mathrm{tor}} \otimes \mathcal{F}^*_{\mathrm{tf}}$ and $\mathcal{F}^* \otimes \mathcal{F}^* \to \mathcal{F}^*_{\mathrm{tf}} \otimes \mathcal{F}^*_{\mathrm{tf}}$.

Proof. By taking filtered direct limits, it suffices to prove those statements with $\mathcal{F}^*$ replaced with a finitely generated subgroup A of $\mathcal{F}^*$ that contains -1. We can then obtain a splitting $A \simeq A_{\mathrm{tor}} \oplus A_{\mathrm{tf}}$ and find that the quotient for A is isomorphic to

$$\frac{A_{\mathrm{tor}} \otimes A_{\mathrm{tor}} \oplus A_{\mathrm{tor}} \otimes A_{\mathrm{tf}} \oplus A_{\mathrm{tf}} \otimes A_{\mathrm{tor}} \oplus A_{\mathrm{tf}} \otimes A_{\mathrm{tf}}}{\langle ((-u) \otimes u, (-u) \otimes c, c \otimes u, c \otimes c) \text{ with } u \text{ in } A_{\mathrm{tor}} \text{ and } c \text{ in } A_{\mathrm{tf}} \rangle}.$$

Our claims follow for A if we prove that the intersection of

$$A_{\mathrm{tor}} \otimes A_{\mathrm{tor}} \oplus A_{\mathrm{tor}} \otimes A_{\mathrm{tf}} \oplus A_{\mathrm{tf}} \otimes A_{\mathrm{tor}} \oplus 0$$

with the group in the denominator equals

$$\langle ((-u) \otimes u, u \otimes c, c \otimes u, 0) \text{ with } u \text{ in } A_{\mathrm{tor}} \text{ and } c \text{ in } A_{\mathrm{tf}} \rangle$$

as the latter is the product

$$\langle (-u) \otimes u \text{ with } u \text{ in } A_{\mathrm{tor}} \rangle \times \langle (v \otimes c, c \otimes v) \text{ with } v \text{ in } A_{\mathrm{tor}} \text{ and } c \text{ in } A_{\mathrm{tf}} \rangle \times \{0\}.$$

From the identity

$$(-uc) \otimes uc - (-c) \otimes c = (-u) \otimes u + u \otimes c + c \otimes u$$

in $A \times A$ it is clear that this intersection contains the given subgroup, so we only have to show it is not larger. For this, choose a basis $b_1, \ldots, b_s$ of A_{tf} and assume that, for some integers m_i, the last position in

$$\sum_i m_i((-u_i) \otimes u_i, (-u_i) \otimes c_i, c_i \otimes u_i, c_i \otimes c_i) \tag{3.6}$$

is trivial. If $a_{i,j}$ is the coefficient of b_j in c_i, then $\sum_i m_i a_{i,j}^2 = 0$ for each j. So each sum $\sum_i m_i a_{i,j}$ is even, the sum

$$\sum_i m_i(-1) \otimes c_i = \sum_j \sum_i m_i a_{i,j}(-1) \otimes b_j$$

is trivial, and in the second position of the element in (3.6) we can replace each $-u_i$ with u_i. □

Remark 3.7. Clearly, Fil_1 is trivial if $\mathcal{F}$ has characteristic 2. It is also trivial if that characteristic is not equal to 2 but $\mathcal{F}^*$ contains an element of order 4: if u in $\mathcal{F}^*$ has order $2m$ with m even, then $u \tilde{\wedge} u = (-1) \tilde{\wedge} u = m(u \tilde{\wedge} u)$ in $\tilde{\wedge}^2 \mathcal{F}^*$, and $\gcd(m-1, 2m) = 1$. Finally, if $\mathcal{F}$ has characteristic not equal to 2, and $\mathcal{F}^*$ does not contain an element of order 4, then by decomposing $\mathcal{F}^*_{\mathrm{tor}}$ into its primary components, one sees that Fil_1 is cyclic of order 2, generated by $(-1) \tilde{\wedge} (-1)$.

Corollary 3.8. *Let F be a number field, n the order of F^*_{tor}, and $c_1, c_2, \ldots$ in F^* such that they give a basis of F^*_{tf}. Let $m = 1$ and $u = 1$ if n is divisible by 4, and $m = 2$ and $u = -1$ otherwise. Then the map*

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} \times \oplus_i \mathbb{Z}/n\mathbb{Z} \times \oplus_{i<j} \mathbb{Z} &\to \tilde{\wedge}^2 F^* \\ (a, (b_i)_i, (b_{i,j})_{i,j}) &\mapsto u^a \tilde{\wedge} u + \sum_i u^{b_i} \tilde{\wedge} c_i + \sum_{i<j} c_i^{b_{i,j}} \tilde{\wedge} c_j \end{aligned}$$

is an isomorphism.

Proof. There is a filtration Fil'_l for $l = 0$, 1, 2 and 3 on the domain by taking the last $3 - l$ positions to be trivial, and under the homomorphism we map Fil'_l to Fil_l as in Proposition 3.5 so it induces a homomorphism $\mathrm{Fil}'_l/\mathrm{Fil}'_{l-1} \to \mathrm{Fil}_l/\mathrm{Fil}_{l-1}$ for $l = 1$, 2 and 3. For $l = 1$ this is an isomorphism by Remark 3.7, and for $l = 2$ and $l = 3$ by Proposition 3.5. We now apply the five-lemma. □

Remark 3.9. (i) One can get finitely many of the c_i in the corollary by taking a basis of the free part of the S-units for a finite set S of primes of the ring of integers $\mathcal{O}$ of k. If one extends S to S', then one can add more c_j in order to obtain a similar basis for the S'-units.

(ii) Given a generator u of F^*_{tor} of order $2l$ and finitely many of the c_i in the corollary, that together generate a subgroup A of F^*, the isomorphism of the corollary becomes explicit on the image of $\tilde{\wedge}^2 A$ by writing everything out in terms of the generators, and using $c_i \tilde{\wedge} c_j + c_j \tilde{\wedge} c_i = 0$ if $i \neq j$, $c_i \tilde{\wedge} c_i = (-1)\tilde{\wedge} c_i$, $u \tilde{\wedge} c_i + c_i \tilde{\wedge} u = 0$, as well as that $u \tilde{\wedge} u$ is equal to $(-1)\tilde{\wedge}(-1)$ if l is odd, and trivial if l is even.

3.2.2. Any field extension $\mathcal{F} \to \mathcal{F}'$ induces a homomorphism from $\tilde{\wedge}^2 \mathcal{F}^*$ to $\tilde{\wedge}^2 (\mathcal{F}')^*$. In the next result we determine the kernel of this map in the case that $\mathcal{F} = \mathbb{Q}$ and $\mathcal{F}'$ is imaginary quadratic. This result will play a important role for Theorem 4.7.

Lemma 3.10. *Let d be a positive square-free integer, and let $k = \mathbb{Q}(\sqrt{-d})$. If $d \neq 1$ then the kernel of the map $\tilde{\wedge}^2 \mathbb{Q}^* \to \tilde{\wedge}^2 k^*$ has order 2, with $(-1)\tilde{\wedge}(-d)$ as non-trivial element. If $d = 1$ then the kernel is non-cyclic of order 4, and is generated by $(-1)\tilde{\wedge}(-1)$ and $(-1)\tilde{\wedge} 2$.*

Proof. The given elements are clearly in the kernel (for $\mathbb{Q}(\sqrt{-1})$ use $2 = \sqrt{-1}(1 - \sqrt{-1})^2$ and that $\sqrt{-1}\tilde{\wedge}\sqrt{-1}$ is trivial by Remark 3.7). In addition, by Proposition 3.5 (or Corollary 3.8 with the prime numbers as c_i) the elements generate a subgroup of the stated order. It is therefore enough to check the size of the kernel.

Clearly, from the description in Proposition 3.5 the kernel is contained in Fil_2 on $\tilde{\wedge}^2 \mathbb{Q}$. Because the map on the Fil_1-pieces is surjective by Remark 3.7, with kernel of order 2 if $d = 1$ and trivial otherwise, we only have to show that the kernel for $\mathrm{Fil}_2/\mathrm{Fil}_1$ has order 2.

For this we use the description of $\mathrm{Fil}_2/\mathrm{Fil}_1$ in Proposition 3.5. If $|k^*_{\mathrm{tor}}| = 2m$ then the kernel corresponds to the kernel of the map $\mathbb{Q}^*_{\mathrm{tf}}/2 \to k^*_{\mathrm{tf}}/2m$ given by raising to the mth power, as -1 is the mth power of a generator of k^*_{tor}. We solve $a^m = u\alpha^{2m}$ with a in $\mathbb{Q}^*$, u in k^*_{tor}, and α in k^*, or, equivalently, $a = v\alpha^2$ for some v in k^*_{tor} as u is an mth power in k^*.

After some calculation, we find that, for $d \neq 1$, a is of the form $\pm b^2$ or $\pm d \cdot b^2$ for some b in $\mathbb{Q}^*$, and for $d = 1$ that it is of the form $\pm b^2$ or $\pm 2b^2$. In either case, this leads to two elements in $\mathbb{Q}^*_{\mathrm{tf}}/2$ that are in the kernel, as required. □

3.3. **Configurations of points, and a modified Bloch group.** In order to be able to use the result of Theorem 3.4 in the geometric construction of elements in the indecomposable K_3-groups of imaginary quadratic fields that we shall make in §4, it is convenient to make technical modifications of well-known constructions of Suslin [43] and of Goncharov [24, p. 73]. As a result, we shall be able to be more precise about torsion in the resulting Bloch groups and some of the homomorphisms involved.

However, in order to be able to take finite non-trivial torsion in stabilisers of points in $\mathbb{P}^1_{\mathcal{F}}$ into account, and to be able to work with groups like $\mathrm{PGL}_2(\mathcal{F})$ instead of $\mathrm{GL}_2(\mathcal{F})$ whenever necessary, we are forced to work in somewhat greater generality.

3.3.1. Let $\mathcal{F}$ be a field, and fix two subgroups $\nu \subseteq \nu'$ of $\mathcal{F}^*$. (Typically, we have in mind $\nu = \{1\}$ or $\{\pm 1\}$, and ν' the torsion subgroup of the units of the ring of algebraic integers in a number field.) Let $\Delta = \mathrm{GL}_2(\mathcal{F})/\nu$. Let $\mathcal{L}$ be the set of orbits for the action of ν' on $\mathcal{F}^2 \setminus \{(0,0)\}$ given by scalar multiplication, which has a natural map to $\mathbb{P}^1_{\mathcal{F}}$. The extreme cases $\nu' = \mathcal{F}^*$ and $\nu' = \{1\}$ give $\mathcal{L} = \mathbb{P}^1_{\mathcal{F}}$ and $\mathcal{L} = \mathcal{F}^2 \setminus \{(0,0)\}$ respectively. For $n \geq 0$ we let $C_n(\mathcal{L})$ be the free abelian group with as generators $(n+1)$-tuples $(l_0, \dots, l_n)$ of elements in $\mathcal{L}$ such that if l_{i_1} and l_{i_2} have the same image in $\mathbb{P}^1_{\mathcal{F}}$ then $l_{i_1} = l_{i_2}$ (see Remark 3.15 for an explanation of this condition). We shall call such a tuple $(l_0, \dots, l_n)$ with all l_i distinct in $\mathcal{L}$ (or equivalently, in $\mathbb{P}^1_{\mathcal{F}}$) non-degenerate, and we shall call it degenerate otherwise. Then Δ acts on $C_n(\mathcal{L})$ as $\nu \subseteq \nu'$, and with the usual boundary map $\mathrm{d}\colon C_n(\mathcal{L}) \to C_{n-1}(\mathcal{L})$ for $n \geq 1$ given by

$$\mathrm{d}((l_0, \dots, l_n)) = \sum_{i=0}^{n} (-1)^i (l_0, \dots, \widehat{l_i}, \dots, l_n),$$

where $\widehat{l_i}$ indicates that the term l_i is omitted, we get a complex

$$\cdots \xrightarrow{\mathrm{d}} C_4(\mathcal{L}) \xrightarrow{\mathrm{d}} C_3(\mathcal{L}) \xrightarrow{\mathrm{d}} C_2(\mathcal{L}) \xrightarrow{\mathrm{d}} C_1(\mathcal{L}) \xrightarrow{\mathrm{d}} C_0(\mathcal{L}) \tag{3.11}$$

of $\mathbb{Z}[\Delta]$-modules.

For three non-zero points p_0, p_1 and p_2 in $\mathcal{F}^2$ with distinct images in $\mathbb{P}^1_{\mathcal{F}}$, we define $\mathrm{cr}_2(p_0, p_1, p_2)$ in $\tilde{\wedge}^2 \mathcal{F}^*$ by the rules:

- $\mathrm{cr}_2(gp_0, gp_1, gp_2) = \mathrm{cr}_2(p_0, p_1, p_2)$ for every g in $\mathrm{GL}_2(\mathcal{F})$;
- $\mathrm{cr}_2((1,0),(0,1),(a,b)) = (a)\tilde{\wedge}(b)$. [1]

Because $\mathrm{cr}_2((0,1),(1,0),(a,b)) = (b)\tilde{\wedge}(a)$ and $\mathrm{cr}_2((1,0),(a,b),(0,1)) = (-ab^{-1})\tilde{\wedge}(b^{-1}) = -(a)\tilde{\wedge}(b)$, we see that cr_2 is alternating. It is also clear that if we scale one of the p_i by λ in ν' then $\mathrm{cr}_2(p_0, p_1, p_2)$ changes by a term $(\lambda)\tilde{\wedge}(c)$ with c in $\mathcal{F}^*$. Let

$$\tilde{\wedge}^2 \mathcal{F}^* / \nu' \tilde{\wedge} \mathcal{F}^* = \frac{\tilde{\wedge}^2 \mathcal{F}^*}{\langle (\lambda)\tilde{\wedge}(c) \text{ with } \lambda \text{ in } \nu' \text{ and } c \text{ in } \mathcal{F}^* \rangle}. \tag{3.12}$$

We then obtain a group homomorphism

$$f_{2,\mathcal{F}}\colon C_2(\mathcal{L}) \to \tilde{\wedge}^2 \mathcal{F}^* / \nu' \tilde{\wedge} \mathcal{F}^*$$

by letting this be trivial on a degenerate generator (l_0, l_1, l_2), and by mapping a non-degenerate generator (l_0, l_1, l_2) to $\mathrm{cr}_2(p_0, p_1, p_2)$ with p_i a point in l_i. (Clearly ν' should be part of the notation, but we suppress it for this map.)

We next define a group homomorphism

$$f_{3,\mathcal{F}}\colon C_3(\mathcal{L}) \to \mathbb{Z}[\mathcal{F}^\flat]$$

as follows. On a degenerate generator (l_0, l_1, l_2, l_3) we let $f_{3,\mathcal{F}}$ be trivial, and we let it map a non-degenerate generator (l_0, l_1, l_2, l_3) to $[\mathrm{cr}_3(\overline{l_0}, \overline{l_1}, \overline{l_2}, \overline{l_3})]$, the generator corresponding to the cross-ratio cr_3 of the images of the points in $\mathbb{P}^1_{\mathcal{F}}$. Recall that cr_3 is defined by rules similar to those for cr_2:

- $\mathrm{cr}_3(g\overline{l_0}, g\overline{l_1}, g\overline{l_2}, g\overline{l_3}) = \mathrm{cr}_3(\overline{l_0}, \overline{l_1}, \overline{l_2}, \overline{l_3})$ for every g in $\mathrm{GL}_2(\mathcal{F})$;
- $\mathrm{cr}_3([1,0],[0,1],[1,1],[x,1]) = x$ for x in $\mathcal{F}^\flat$.

[1] Goncharov in [24, §3] maps this to $(-1)\tilde{\wedge}(-1) + (b)\tilde{\wedge}(a)$.

Remark 3.13. From the $\mathrm{GL}_2(\mathcal{F})$-equivariance of cr_3 one sees by a direct calculation that, for l_0, l_1, l_2, l_3 different non-zero points in $\mathcal{F}^2$,

$$\mathrm{cr}_3(\overline{l_0}, \overline{l_1}, \overline{l_2}, \overline{l_3}) = \frac{\det(\begin{bmatrix} l_1 & l_3 \end{bmatrix}) \det(\begin{bmatrix} l_2 & l_4 \end{bmatrix})}{\det(\begin{bmatrix} l_1 & l_4 \end{bmatrix}) \det(\begin{bmatrix} l_2 & l_3 \end{bmatrix})}.$$

As is well known, from this, or by a direct calculation, we see that permuting the four points may result in the following related possibilities for a cross ratio: x, $1 - x^{-1}$, $(1-x)^{-1}$ for even permutations, and $1 - x$, x^{-1}, $(1 - x^{-1})^{-1}$ for odd ones, with the subgroup V_4 of S_4 acting trivially.

3.3.2. In the next result we consider the homomorphism

$$\delta^{\nu'}_{2,\mathcal{F}} \colon \mathbb{Z}[\mathcal{F}^\flat] \to \tilde{\wedge}^2 \mathcal{F}^* / \nu' \tilde{\wedge} \mathcal{F}^*$$

that sends each element $[x]$ for x in $\mathcal{F}^\flat$ to the class of $(1-x)\tilde{\wedge}x$. If ν' is trivial then this is still the map $\delta_{2,\mathcal{F}}$ of (3.2).

Lemma 3.14. *The following diagram commutes.*

$$\begin{array}{ccc} C_3(\mathcal{L}) & \xrightarrow{\mathrm{d}} & C_2(\mathcal{L}) \\ \Big\downarrow f_{3,\mathcal{F}} & & \Big\downarrow f_{2,\mathcal{F}} \\ \mathbb{Z}[\mathcal{F}^\flat] & \xrightarrow{\delta^{\nu'}_{2,\mathcal{F}}} & \tilde{\wedge}^2 \mathcal{F}^* / \nu' \tilde{\wedge} \mathcal{F}^* \end{array}$$

Proof. It suffices to check commutativity for each generating element (l_0, l_1, l_2, l_3) of $C_3(\mathcal{L})$.

If (l_0, l_1, l_2, l_3) is non-degenerate, this follows by an explicit computation. Specifically, by using the $\mathrm{GL}_2(\mathcal{F})$-invariance of both $f_{3,\mathcal{F}}$ and $f_{2,\mathcal{F}}$ and the $\mathrm{GL}_2(\mathcal{F})$-equivariance of d one can assume that (l_0, l_1, l_2, l_3) are the classes of $(a, 0)$, $(0, b)$, $(1, 1)$ and (xc, c) in $\mathcal{L}$ for some and a, b and c in $\mathcal{F}^*$ and x in $\mathcal{F}^\flat$, which results in $[x]$ in $\mathbb{Z}[\mathcal{F}^\flat]$ under $f_{3,\mathcal{F}}$ and the class of $(1-x)\tilde{\wedge}(x)$ under $f_{2,\mathcal{F}} \circ \mathrm{d}$.

For a degenerate tuple (l_0, l_1, l_2, l_3) the commutativity is obvious if $\{l_0, l_1, l_2, l_3\}$ has at most two elements as then $f_{2,\mathcal{F}}$ is trivial on every term in $\mathrm{d}((l_0, l_1, l_2, l_3))$.

If $\{l_0, l_1, l_2, l_3\}$ consists of three classes A, B and C with A occurring twice among l_0, l_1, l_2 and l_3, then up to permuting B and C the six possibilities for (l_0, l_1, l_2, l_3) are (A, A, B, C), (A, B, A, C), (A, B, C, A), (B, A, A, C), (B, A, C, A) and (B, C, A, A). After cancellation of identical terms with opposite signs in $\mathrm{d}((l_0, l_1, l_2, l_3))$, we see that commutativity follows because $f_{2,\mathcal{F}}$ is alternating. □

Remark 3.15. The argument used to prove Lemma 3.14 provides the motivation for considering only tuples $(l_0, \ldots, l_n)$ of elements in $\mathcal{L}$ such that if l_{i_1} and l_{i_2} have the same image in $\mathbb{P}^1_{\mathcal{F}}$ then $l_{i_1} = l_{i_2}$. It seems reasonable to define $f_{2,\mathcal{F}}$ and $f_{3,\mathcal{F}}$ to be trivial on tuples for which some points have the same image in $\mathbb{P}^1_{\mathcal{F}}$. Starting with such a tuple (A, A', B, C) where A and A' have the same image, but A, B and C have different images, we require $f_{2,\mathcal{F}}$ takes the same value on (A, B, C) and (A', B, C) and so must limit the amount of scaling between A and A' to ν'.

3.3.3. We now set

$$\overline{\mathfrak{p}}(\mathcal{F}) := \frac{\mathbb{Z}[\mathcal{F}^\flat]}{(f_{3,\mathcal{F}} \circ \mathrm{d})(C_4(\mathcal{L}))}.$$

Then the diagram in Lemma 3.14 induces a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\mathrm{d}} & C_4(\mathcal{L}) & \xrightarrow{\mathrm{d}} & C_3(\mathcal{L}) & \xrightarrow{\mathrm{d}} & C_2(\mathcal{L}) & \xrightarrow{\mathrm{d}} & C_1(\mathcal{L}) & \xrightarrow{\mathrm{d}} & C_0(\mathcal{L}) \\ & & \downarrow & & \downarrow{\scriptstyle f_{3,\mathcal{F}}} & & \downarrow{\scriptstyle f_{2,\mathcal{F}}} & & \downarrow & & \\ & & 0 & \longrightarrow & \overline{\mathfrak{p}}(\mathcal{F}) & \xrightarrow{\partial^{\nu'}_{2,\mathcal{F}}} & \tilde{\wedge}^2\mathcal{F}^*/\nu'\tilde{\wedge}\mathcal{F}^* & \longrightarrow & 0 & & \end{array} \tag{3.16}$$

in which $\partial^{\nu'}_{2,\mathcal{F}}$ denotes the map induced by $\delta^{\nu'}_{2,\mathcal{F}}$. (If ν' is trivial then we shall use the notation $\partial_{2,L}$ for the induced map.) We observe that we could take $\mathrm{GL}_2(\mathcal{F})$-coinvariants in the top row because of the properties of $f_{3,\mathcal{F}}$ and $f_{2,\mathcal{F}}$. In particular, $f_{3,\mathcal{F}}$ induces a homomorphism $H_3(C_\bullet(\mathcal{L})_{\mathrm{GL}_2(\mathcal{F})}) \to \overline{B}(\mathcal{F})_{\nu'}$, where we set

$$\overline{B}(\mathcal{F})_{\nu'} := \ker(\partial^{\nu'}_{2,\mathcal{F}}) .$$

We shall denote this latter group more simply as $\overline{B}(\mathcal{F})$ if ν' is trivial.

The following result provides an explicit description of the relations in $\overline{\mathfrak{p}}(\mathcal{F})$ and will be very useful in later arguments.

Lemma 3.17. *The subgroup $(f_{3,\mathcal{F}} \circ \mathrm{d})(C_4(\mathcal{L}))$ of $\mathbb{Z}[\mathcal{F}^\flat]$ is generated by all elements of the form*

$$[x] - [y] + [y/x] - [(1-y)/(1-x)] + [(1-y^{-1})/(1-x^{-1})] \tag{3.18}$$

for $x \neq y$ in $\mathcal{F}^\flat$, and

$$[x] + [x^{-1}] \text{ and } [y] + [1-y] \tag{3.19}$$

for x and y in $\mathcal{F}^\flat$.

Proof. We note first that for each non-degenerate generator $(l_0, \ldots, l_4)$ of $C_4(\mathcal{L})$ one has

$$(f_{3,\mathcal{F}} \circ \mathrm{d})((l_0, \ldots, l_4)) = \sum_{i=1}^{5} (-1)^i \,\mathrm{cr}_3(\overline{l_0}, \ldots, \widehat{\overline{l_i}}, \ldots, \overline{l_4}),$$

where $\overline{l_0}, \ldots, \overline{l_4}$ are distinct points in $\mathbb{P}^1_{\mathcal{F}}$ and $\widehat{\overline{l_i}}$ indicates that the term $\overline{l_i}$ is omitted.

In view of the invariance of cr_3 under the action of $\mathrm{GL}_2(\mathcal{F})$, and the fact that for cr_3 we can use points in $\mathbb{P}^1_{\mathcal{F}}$, we may assume the points are $(1,0)$, $(0,1)$, $(1,1)$, $(x,1)$ and $(y,1)$ for $x \neq y$ in $\mathcal{F}^\flat$. This then under $f_{3,\mathcal{F}} \circ \mathrm{d}$ yields the element (3.18).

Let now $(l_0, \ldots, l_4)$ be a degenerate generator. Then its image under $f_{3,\mathcal{F}} \circ \mathrm{d}$ is trivial if $\{l_0, \ldots, l_4\}$ has at most three elements since then all the terms in $\mathrm{d}((l_0, \ldots, l_4))$ are degenerate. On the other hand, if $\{l_0, \ldots, l_4\}$ has four elements, then after cancelling possible identical terms in $\mathrm{d}((l_0, \ldots, l_4))$ and applying cr_3 to the result we see that it is of the form

$$[\mathrm{cr}_3(\overline{m_1}, \ldots, \overline{m_4})] - \mathrm{sgn}(\sigma)[\mathrm{cr}_3(\overline{m_{\sigma(1)}}, \ldots, \overline{m_{\sigma(4)}})]$$

for a permutation σ in S_4 with sign $\mathrm{sgn}(\sigma)$, and four distinct points $\overline{m_i}$ in $\mathbb{P}^1_{\mathcal{F}}$. The subgroup generated by these images coincides with the subgroup generated by the terms (3.19). (This shows, in particular, that the map $f_{3,\mathcal{F}}$ is alternating.) □

Remark 3.20. If ν' is finite of order a, then multiplying an element in $\overline{B}(\mathcal{F})_{\nu'}$ in the bottom row of (3.16) by a gives an element in $\overline{B}(\mathcal{F})$.

Remark 3.21. The map $\mathbb{Z}[F^\flat] \to \mathbb{R}(1)$ in Theorem 3.4(iii), mapping $[x]$ to $iD(\sigma(x))$ for an embedding $\sigma\colon F \to \mathbb{C}$, induces a map $\mathbb{D}_\sigma\colon \overline{\mathfrak{p}}(F) \to \mathbb{R}(1)$ because the relations in (3.18) and (3.19) are matched by (3.3).

3.3.4. We next show that the group $\overline{B}(\mathcal{F})$ defined above is naturally isomorphic to a quotient of the 'Bloch group' $B(\mathcal{F})$ that is defined by Suslin in [43]. This result motivates us to regard $\overline{B}(\mathcal{F})$ as a modified Bloch group (and therefore explains our choice of notation). In fact, we shall establish the precise relation between our groups $\overline{B}(\mathcal{F})$ and $\overline{\mathfrak{p}}(\mathcal{F})$ and the corresponding groups $B(\mathcal{F})$ and $\mathfrak{p}(\mathcal{F})$. Before proving our statements, we first recall the group defined in [43].

For an infinite field $\mathcal{F}$, Suslin considers the group

$$\mathfrak{p}(\mathcal{F}) = \frac{\mathbb{Z}[\mathcal{F}^\flat]}{\langle [x] - [y] + [\frac{y}{x}] + [\frac{1-x}{1-y}] - [\frac{1-x^{-1}}{1-y^{-1}}] \text{ with } x, y \text{ in } \mathcal{F}^\flat, x \neq y \rangle}.$$

He then defines $B(\mathcal{F})$ to be the kernel of the homomorphism

$$\begin{aligned} \mathfrak{p}(\mathcal{F}) &\to (\mathcal{F}^* \otimes \mathcal{F}^*)_\sigma \\ [x] &\mapsto x \overset{\sigma}{\otimes} (1-x)\,, \end{aligned}$$

where we set

$$(\mathcal{F}^* \otimes \mathcal{F}^*)_\sigma := \frac{\mathcal{F}^* \otimes \mathcal{F}^*}{\langle x \otimes y + y \otimes x \text{ with } x, y \text{ in } \mathcal{F}^* \rangle}$$

and write $a \overset{\sigma}{\otimes} b$ for the class in the quotient of an element $a \otimes b$.

We further recall from loc. cit. that Suslin proves the existence of a canonical short exact sequence of the form

$$0 \to \mathrm{Tor}(\mathcal{F}^*, \mathcal{F}^*)^\sim \to K_3(\mathcal{F})^{\mathrm{ind}} \to B(\mathcal{F}) \to 0, \tag{3.22}$$

where $\mathrm{Tor}(\mathcal{F}^*, \mathcal{F}^*)^\sim$ denotes the unique non-trivial extension of $\mathrm{Tor}(\mathcal{F}^*, \mathcal{F}^*)$ by $\mathbb{Z}/2\mathbb{Z}$ if $\mathcal{F}$ has characteristic different from 2, and denotes $\mathrm{Tor}(\mathcal{F}^*, \mathcal{F}^*)$ otherwise, and $K_3(\mathcal{F})^{\mathrm{ind}}$ is the cokernel of the natural homomorphism from the Milnor K-group $K_3^M(\mathcal{F})$ to $K_3(\mathcal{F})$.

We also recall that the element $c_\mathcal{F} = [x] + [1-x]$ of $B(\mathcal{F})$ is independent of x in $\mathcal{F}^\flat$ and has order dividing 6 [43, Lem. 1.3, 1.5] and that the order of $c_\mathbb{Q}$ in $B(\mathbb{Q})$ is equal to 6 [43, Prop. 1.1].

3.3.5. Lemma 3.17 implies that the group $\overline{\mathfrak{p}}(\mathcal{F})$ defined above is obtained from $\mathfrak{p}(\mathcal{F})$ by quotienting out by the subgroup generated by all elements of the form $[x] + [x^{-1}]$ with x in $\mathcal{F}^\flat$ and $[y] + [1-y]$ with y in $\mathcal{F}^\flat$.

Since the latter elements generate the same group as does the element $c_\mathcal{F}$ defined above we are motivated to consider the following short exact sequence of complexes (with vertical differentials)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle [x] + [x^{-1}] \text{ with } x \text{ in } \mathcal{F}^\flat \rangle & \longrightarrow & \mathfrak{p}(\mathcal{F})/\langle c_\mathcal{F} \rangle & \longrightarrow & \overline{\mathfrak{p}}(\mathcal{F}) & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \downarrow \partial_{2,\mathcal{F}} & & \\ 0 & \longrightarrow & \langle x \overset{\sigma}{\otimes} (-x) \text{ with } x \text{ in } \mathcal{F}^\flat \rangle & \longrightarrow & (\mathcal{F}^* \otimes \mathcal{F}^*)_\sigma & \longrightarrow & \tilde{\wedge}^2 \mathcal{F}^* & \longrightarrow & 0. \end{array}$$

Theorem 3.23. *If $\mathcal{F}$ is infinite, then the homomorphism f in the above diagram is bijective, and, in particular, the diagram induces an isomorphism $B(\mathcal{F})/\langle c_\mathcal{F} \rangle \to \overline{B}(\mathcal{F})$.*

Proof. A direct calculation shows that f maps the class of $[x] + [x^{-1}]$ to $x \overset{\sigma}{\otimes} (-x)$, so f is surjective.

In order to show that f is injective, recall that by [43, Lem. 1.2], the map $\mathcal{F}^* \to \mathfrak{p}(\mathcal{F})$ sending x to $[x]+[x^{-1}]$ if $x \neq 1$ and 1 to 0, is a homomorphism, and that $(\mathcal{F}^*)^2$ is in its kernel. We shall consider its composition with the quotient map to $\mathfrak{p}(\mathcal{F})/\langle c_{\mathcal{F}}\rangle$, giving a surjective homomorphism

$$g\colon \mathcal{F}^* \to \langle [x] + [x^{-1}] \text{ with } x \text{ in } \mathcal{F}^\flat \rangle\,,$$

with the target in $\mathfrak{p}(\mathcal{F})/\langle c_{\mathcal{F}}\rangle$. If -1 is a square then we already know that $[-1] + [-1] = 0$ in $\mathfrak{p}(\mathcal{F})$. If -1 is not a square then $2 \neq 0$, so $2[-1] = 2c_{\mathcal{F}} - 2[2] = 2c_{\mathcal{F}} + 2[\frac{1}{2}] = 3c_{\mathcal{F}}$ in $\mathfrak{p}(\mathcal{F})$ (cf. [43, Lem. 1.4]). In either case, we have that $\{\pm 1\} \cdot (\mathcal{F}^*)^2 \subseteq \ker(g)$, and that $\operatorname{im}(g)$ is the subgroup generated by the classes of $[x]+[x^{-1}]$ with x in $\mathcal{F}^\flat$. We also want to consider $\ker(f \circ g)$. For this, we fix a basis $\mathcal{B}$ of $\mathcal{F}^*/(\mathcal{F}^*)^2$ as $\mathbb{F}_2$-vector space, making sure to include -1 in $\mathcal{B}$ if -1 is not a square in $\mathcal{F}^*$. For b in $\mathcal{B}$, the homomorphism $\mathcal{F}^*/(\mathcal{F}^*)^2 \to \mathbb{F}_2 \cdot b \simeq \mathbb{F}_2$ obtained from the projection onto $\mathbb{F}_2 \cdot b$ can be applied twice in the tensor product in order to give a composite homomorphism $\mathcal{F}^* \otimes \mathcal{F}^* \to \mathcal{F}^*/(\mathcal{F}^*)^2 \otimes \mathcal{F}^*/(\mathcal{F}^*)^2 \to \mathbb{F}_2 \otimes \mathbb{F}_2 \simeq \mathbb{F}_2$. This induces a homomorphism $h_b\colon (\mathcal{F}^* \otimes \mathcal{F}^*)_\sigma \to \mathbb{F}_2$, mapping $x \overset{\sigma}{\otimes} y$ to the product of the coefficients of b in the classes of x and y in $\mathcal{F}^*/(\mathcal{F}^*)^2$. If x in $\mathcal{F}^*$ is in $\ker(f \circ g)$, then $h_b(x \overset{\sigma}{\otimes} (-x)) = 0$ for all b. If -1 is a square, this means x is a square. If -1 is not a square, then x or $-x$ must be a square. In either case, it follows that $\ker(f \circ g) \subseteq \{\pm 1\} \cdot (\mathcal{F}^*)^2$. Because $\{\pm 1\} \cdot (\mathcal{F}^*)^2 \subseteq \ker(g)$ and g is surjective, it follows that f is injective.

So f is an isomorphism. The isomorphism $B(\mathcal{F})/\langle c_{\mathcal{F}}\rangle \to \overline{B}(\mathcal{F})$ now follows from the snake lemma. □

Remark 3.24. Note that in the proof above, it also follows that $\ker(g) = \{\pm 1\}(\mathcal{F}^*)^2$. So g induces an isomorphism from $\mathcal{F}^*/\{\pm 1\} \cdot (\mathcal{F}^*)^2$ to the subgroup of $\mathfrak{p}(\mathcal{F})/\langle c_F\rangle$ generated by the classes of $[x]+[x^{-1}]$ with x in $\mathcal{F}^\flat$, given by mapping the class of x to the class of $[x]+[x^{-1}]$. And $f \circ g$ induces an isomorphism from $\mathcal{F}^*/\{\pm 1\} \cdot (\mathcal{F}^*)^2$ to the subgroup of $(\mathcal{F}^* \otimes \mathcal{F}^*)_\sigma$ generated by the $x \overset{\sigma}{\otimes} (-x)$, mapping the class of x to the class of $x \overset{\sigma}{\otimes} (-x)$.

3.3.6. We can now state the main result of this section. This result concerns the map $\varphi_{\mathcal{F}}$ in Theorem 3.4.

Theorem 3.25. *$\varphi_{\mathcal{F}}$ induces a homomorphism $\psi_{\mathcal{F}}\colon \overline{B}(\mathcal{F}) \to K_3(\mathcal{F})_{\mathrm{tf}}^{\mathrm{ind}}$.*

Proof. In view of Lemma 3.17, it is enough to show that $\varphi_{\mathcal{F}}$ is trivial on all elements of the form (3.18) and (3.19).

If $\mathcal{F}$ is a number field F then this follows from Theorem 3.4(iii), (3.3), and Borel's theorem, Theorem 2.1, by letting σ run through all embeddings of F into $\mathbb{C}$.

In order to see that it holds for all fields $\mathcal{F}$ as in Theorem 3.4, we can tensor with $\mathbb{Q}$, in which case the construction underlying the construction of the map $\varphi_{\mathcal{F}}$ in Theorem 3.4 is the simplest case of the constructions that are made by the second author in [17].

One can then verify that the elements in (3.19) and (3.18) are trivial by working over $\mathbb{Z}[x, x^{-1}, (1-x)^{-1}]$ or $\mathbb{Z}[x, y, (1-x)^{-1}, (1-y)^{-1}, (x-y)^{-1}]$ as the base schemes, along the lines of the proof of [17, Prop. 6.1] where the base scheme can be taken to be $\mathbb{Z}[x, x^{-1}, (1-x)^{-1}]$. We leave the precise details of this argument to an interested reader. □

Remark 3.26. In this remark we assume ν' to be trivial and explain the advantages to our approach of the definitions that we have adopted in comparison to those that are used by Goncharov in [24].

To be specific, we recall that in (3.8) of loc. cit., a key role is played by the boundary map

$$\mathbb{Z}[\mathcal{F}^\flat] \to \frac{\mathcal{F}^* \otimes \mathcal{F}^*}{\langle a \otimes b + b \otimes a \text{ with } a, b \text{ in } \mathcal{F}^* \rangle} \tag{3.27}$$

that sends a generator $[x]$ to the class of $x \otimes (1-x)$. Our group $\tilde{\wedge}^2 \mathcal{F}^*$ is a quotient of the target group here (cf. the diagram just before Theorem 3.23), and $\delta_{2,\mathcal{F}}$ maps $[x]$ to the inverse of the image of the class of $x \otimes (1-x)$ in $\tilde{\wedge}^2 \mathcal{F}^*$.

Now the map that Goncharov constructs from non-degenerate triples of non-zero points in $\mathcal{F}^2$ to the right hand side of (3.27) is not itself $\mathrm{GL}_2(\mathcal{F})$-equivariant since letting a matrix with determinant c act changes the result by the class of $c \otimes (-c)$.

In addition, the calculation with the four points $(a, 0)$, $(0, b)$, $(1, 1)$ and (xc, c) in the proof of Lemma 3.14 would similarly result in the class in the right hand side of (3.27) of the element $x \otimes (1-x) + c \otimes (-c)$ which is not what one wants.

Whilst these problems could be simply resolved by multiplying any of the relevant maps by a factor of two, this would in the end lead either to a smaller subgroup of $K_3(\mathcal{F})^{\mathrm{ind}}_{\mathrm{tf}}$ if we multiply $f_{3,\mathcal{F}}$ by 2, or a (new) Bloch group that is too large (if we multiply Goncharov's boundary map by 2). It is therefore better for us to avoid the problem by replacing the right hand side of (3.27) as the target of the boundary map by its quotient $\tilde{\wedge}^2 \mathcal{F}^*$.

On the other hand, the elements of the form $[x] + [x^{-1}]$ that are in the kernel of $\delta_{2,\mathcal{F}}$ could then result in a potentially large and undesired subgroup in the kernel of the boundary map, even modulo the 5-term relations (3.18) (see Theorem 3.23 and its proof). To avoid this possible problem we have also imposed the relations (3.19) when defining $\overline{\mathfrak{p}}(k)$ by working with degenerate configurations.

3.4. **Torsion elements in Bloch groups.** In this section we study the torsion subgroup of the modified Bloch group $\overline{B}(F)$ of a number field F by means of a comparison with the Bloch group $B(F)$ defined by Suslin (and recalled in §3.3.4).

In this way, we find that $\overline{B}(F)$ is torsion free if F is equal to either $\mathbb{Q}$, or to an imaginary quadratic number field or is generated over $\mathbb{Q}$ by a root of unity (of any given order). In the case of imaginary quadratic fields, this fact will then play an important role in § 4.

3.4.1. For the sake of simplicity, we formulate and prove the next result only for number fields.

In its statement, if p is a prime number, then we denote the p-primary torsion subgroup of a finitely generated abelian group by means of the subscript p. Because all torsion groups here are finite and cyclic, this determines their structures.

Proposition 3.28. *Let F be a number field. For a prime p, let p^s be the number of p-power roots of unity in F, and let r be the largest integer such that the maximal totally real subfield $\mathbb{Q}(\mu_{p^r})^+$ of $\mathbb{Q}(\mu_{p^r})$ is contained in F. Then the orders of the p-power torsion subgroups in the various groups are as follows.*

prime	$\lvert\mathrm{Tor}(F^*,F^*)_p^\sim\rvert$	$\lvert K_3(F)_p^{\mathrm{ind}}\rvert$	$\lvert B(F)_p\rvert$	$\lvert\overline{B}(F)_p\rvert$	*condition*
$p \geq 5$	p^s	p^r	p^r	p^r	$\zeta_p \notin F$
			1	1	$\zeta_p \in F$
$p = 3$	3^s	3^r	3^r	3^{r-1}	$\zeta_3 \notin F$
			1	1	$\zeta_3 \in F$
$p = 2$	2^{s+1}	2^{r+1}	2^{r-1}	2^{r-2}	$\zeta_4 \notin F$
			1	1	$\zeta_4 \in F$

(*Note* $r \geq 2$ *and* $s \geq 1$ *if* $p = 2$*, and* $r \geq 1$ *if* $p = 3$.)

Proof. We compute $|K_3(F)_p^{\mathrm{ind}}|$ (which is faster than using [50, Chap. IV, Prop. 2.2 and 2.3]).

Let $A \subseteq \mathbb{Z}_p^*$ be the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ in $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_p^*$. Then there are identifications

$$K_3(F)_p^{\mathrm{ind}} \simeq H^0(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \mathbb{Q}_p(2)/\mathbb{Z}_p(2)) \simeq \cap_{a\in A} \ker(\mathbb{Q}_p/\mathbb{Z}_p \overset{a^2-1}{\to} \mathbb{Q}_p/\mathbb{Z}_p),$$

where the first follows from [35, Cor. 4.6] and the second is clear.

We assume for the moment that $p \neq 2$. Then $r = 0$ is equivalent with $A \not\subseteq \{\pm 1\}\cdot(1+p\mathbb{Z}_p)$, so some a^2-1 is in $\mathbb{Z}_p^*$ and the resulting kernel is trivial. For $r \geq 1$, we have $A \subseteq \{\pm1\}\cdot(1+p^r\mathbb{Z}_p)$ but $A \not\subseteq \{\pm1\}\cdot(1+p^{r+1}\mathbb{Z}_p)$. Then $1+p^r\mathbb{Z}_p \subseteq A$ because p is odd, hence $A^2 = 1+p^r\mathbb{Z}_p$ and the statement is clear.

To deal with the case $p = 2$ we note that A is the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ in $\mathrm{Gal}(\mathbb{Q}(\mu_{2^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_2^* = \{\pm1\}\cdot(1+4\mathbb{Z}_2)$. Then $A \subseteq \{\pm1\}\cdot(1+2^r\mathbb{Z}_2)$ but $A \not\subseteq \{\pm1\}\cdot(1+2^{r+1}\mathbb{Z}_2)$, for some $r \geq 2$. In this case, A contains an element of $\{\pm1\}\cdot(1+2^r\mathbb{Z}_2^*)$, and $A^2 = 1+2^{r+1}\mathbb{Z}_2$, from which the statement follows.

We always have $r \geq s$. If ζ_{2p} is in F then $r = s$ because $\mathbb{Q}(\zeta_{p^r})^+(\zeta_{2p}) = \mathbb{Q}(\zeta_{p^r})$. If ζ_{2p} is not in F, then $s = 0$ for $p \neq 2$, and $s = 1$ for $p = 2$. The entries for $|B(F)_p|$ are now immediate from (3.22). From this, we recover that $B(\mathbb{Q})$ has order 6, so is generated by $c_{\mathbb{Q}}$. Then we can compare the sequences (3.22) for the field $\mathbb{Q}$ and for F. Using that $K_3(F)_{\mathrm{tor}}^{\mathrm{ind}}$ is cyclic, and that $c_{\mathbb{Q}}$ maps to c_F under the injection $K_3(\mathbb{Q})_{\mathrm{tor}}^{\mathrm{ind}} \to K_3(F)_{\mathrm{tor}}^{\mathrm{ind}}$, it follows that $3c_F$ has order 2 if and only if $\zeta_4 \notin F$, and that $2c_F$ has order 3 if and only if $\zeta_3 \notin F$: in order to have those orders, we must have $|\mathrm{Tor}(F^*,F^*)_p^\sim| = |\mathrm{Tor}(\mathbb{Q}^*,\mathbb{Q}^*)_p^\sim|$ for $p = 2$ or $p = 3$ respectively. (Cf. [43, Lem. 1.5].) This gives the entries for $|\overline{B}(F)_p| = |(B(F)/\langle c_F\rangle)_p|$. □

If F is a number field, and p a prime number, then combining Theorem 3.23 and Proposition 3.28 gives a precise statement when $\overline{B}(F)$ has no p-torsion.

Theorem 3.29. *Let* F *be a number field. Then, for a given prime number* p*, the group* $\overline{B}(F)$ *has no* p*-torsion if and only if the following condition is satisfied:*

- *if* $p \geq 5$*, then either* $\mathbb{Q}(\zeta_p)^+ \not\subseteq F$ *or* $\mathbb{Q}(\zeta_p) \subseteq F$*;*
- *if* $p = 3$*, then either* $\mathbb{Q}(\zeta_9)^+ \not\subseteq F$ *or* $\mathbb{Q}(\zeta_3) \subseteq F$*;*
- *if* $p = 2$*, then either* $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\zeta_8)^+ \not\subseteq F$ *or* $\mathbb{Q}(\zeta_4) \subseteq F$*.*

Proof. We first state when $\overline{B}(F)$ has non-trivial p-torsion, as this is immediate, using Theorem 3.23: with notation as in Proposition 3.28, this is the case if and only if

- $r \geq 1$ and ζ_p is not in F, when $p \geq 5$;
- $r \geq 2$ and ζ_3 is not in F, when $p = 3$;

- $r \geq 3$ and ζ_4 is not in F when $p = 2$.

Upon negating this statement for each prime p one obtains the claimed result. □

Corollary 3.30. *We have that*

(i) $\overline{B}(\mathbb{Q})$ *is trivial;*
(ii) $\overline{B}(F) \simeq \mathbb{Z}^{[F:\mathbb{Q}]/2}$ *if* $F = \mathbb{Q}(\zeta_N)$ *with* $N \geq 3$*;*
(iii) $\overline{B}(F) \simeq \mathbb{Z}$ *if* F *is an imaginary quadratic field.*

In addition, for each of these fields the composition of the maps $\psi_F\colon \overline{B}(F) \to K_3(F)^{\mathrm{ind}}_{\mathrm{tf}}$ *and* $K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} \to \prod_\sigma \mathbb{R}(1)$*, where* σ *runs through the places of* F*, is injective.*

Proof. We first let F be any number field. Then $\overline{B}(F)$ is a finitely generated abelian group of the same rank as $K_3(F)$ by (3.22) and Theorem 3.23, hence, by Theorem 3.4(ii), the kernel of ψ_F is the torsion subgroup of $\overline{B}(F)$. Because of the behaviour of the regulator with respect to complex conjugation, in Theorem 2.1(iii) we only have to consider all places of F, not all embeddings into $\mathbb{C}$.

It therefore suffices to check that for the number fields listed in the corollary, $\overline{B}(F)$ has no p-torsion for every prime number p. But this follows from Theorem 3.29. □

3.4.2. We conclude this subsection with some statements on the torsion in $\overline{\mathfrak{p}}(F)$ and its behaviour under field extensions, but for simplicity, we do this only for $F = \mathbb{Q}$ or imaginary quadratic.

Proposition 3.31.

(i) The torsion subgroup of $\overline{\mathfrak{p}}(\mathbb{Q})$ *has order* 2 *and is generated by* [2]*.*
(ii) If k *is an imaginary quadratic number field, then the natural map* $\overline{\mathfrak{p}}(\mathbb{Q}) \to \overline{\mathfrak{p}}(k)$ *and its composition with* $\partial_{2,k}$ *are injective when* $k \neq \mathbb{Q}(\sqrt{-1})$*, but for* $k = \mathbb{Q}(\sqrt{-1})$ *both kernels are generated by* [2]*.*

Proof. To prove claim (i) we note Corollary 3.30 implies that $\overline{\mathfrak{p}}(\mathbb{Q})$ injects into $\tilde{\wedge}^2\mathbb{Q}^*$. We also know from Proposition 3.5 (or Corollary 3.8) that the natural map $\langle -1 \rangle \otimes \mathbb{Q}^* \to \tilde{\wedge}^2\mathbb{Q}^*$ gives an isomorphism with the torsion subgroup of the latter.

So we want to compute the kernel of the natural map $\langle -1 \rangle \otimes \mathbb{Q}^* \to K_2(\mathbb{Q})$. Using the tame symbol, and the fact that $\{-1,-1\}$ is non-trivial in $K_2(\mathbb{Q})$, one sees that this kernel is cyclic of order 2, generated by $(-1)\tilde{\wedge}2 = \partial_{2,\mathbb{Q}}([2])$. And $0 = [\frac{1}{2}] + [1 - \frac{1}{2}] = 2[\frac{1}{2}] = -2[2]$.

Turning to claim (ii), we note that the kernel of the composition by Corollary 3.30(i) under $\partial_{2,\mathbb{Q}}$ must inject into the kernel of $\tilde{\wedge}^2\mathbb{Q}^* \to \tilde{\wedge}^2 k^*$, which we computed in Lemma 3.10. In particular, as this kernel is a torsion group, we see from Corollary 3.30(ii) that the kernel of the composition and that of $\overline{\mathfrak{p}}(\mathbb{Q}) \to \overline{\mathfrak{p}}(k)$ coincide as the torsion of $\overline{\mathfrak{p}}(k)$ injects into $\tilde{\wedge}^2 k^*$ under $\partial_{2,k}$.

In addition, by claim (i), those kernels are either trivial or generated by [2]. They contain [2] if and only if $(-1)\tilde{\wedge}2$ is in the kernel of $\tilde{\wedge}^2\mathbb{Q}^* \to \tilde{\wedge}^2 k^*$, which by Lemma 3.10 holds if and only if $k = \mathbb{Q}(\sqrt{-1})$.

This completes the proof. □

Remark 3.32. Note that $[2] = 0$ in $\overline{\mathfrak{p}}(\mathbb{Q}(\sqrt{-1}))$ follows explicitly from (3.18) with $x = \sqrt{-1}$, $y = -\sqrt{-1}$, which gives $[-1] = 0$, as $[2] = -[-1]$ by (3.19).

3.5. **A conjectural link between the groups of Bloch and Suslin.** If $\mathcal{F}$ is an infinite field, then (3.22) gives an isomorphism $K_3(\mathcal{F})^{\mathrm{ind}}_{\mathrm{tf}} \overset{\cong}{\to} B(\mathcal{F})_{\mathrm{tf}}$ and Theorem 3.23 gives an isomorphism $B(\mathcal{F})_{\mathrm{tf}} \overset{\cong}{\to} \overline{B}(\mathcal{F})$.

By Theorem 3.25, one also knows that the homomorphism $\varphi_{\mathcal{F}}$ in Theorem 3.4 induces a homomorphism of the form $\psi_{\mathcal{F}}: \overline{B}(\mathcal{F}) \to K_3(\mathcal{F})^{\mathrm{ind}}_{\mathrm{tf}}$. This in turn induces a homomorphism

$$\psi_{\mathcal{F},\mathrm{tf}}: \overline{B}(\mathcal{F})_{\mathrm{tf}} \to K_3(\mathcal{F})^{\mathrm{ind}}_{\mathrm{tf}},$$

thereby allowing us to form the composition

$$K_3(\mathcal{F})^{\mathrm{ind}}_{\mathrm{tf}} \overset{\cong}{\to} B(\mathcal{F})_{\mathrm{tf}} \overset{\cong}{\to} \overline{B}(\mathcal{F})_{\mathrm{tf}} \xrightarrow{\psi_{\mathcal{F},\mathrm{tf}}} \mathrm{im}(\psi_{\mathcal{F},\mathrm{tf}}) \subseteq K_3(\mathcal{F})^{\mathrm{ind}}_{\mathrm{tf}}\,. \tag{3.33}$$

The group $\mathrm{im}(\psi_{\mathcal{F},\mathrm{tf}})$ that occurs here is essentially the same, modulo torsion, as the group that was originally defined by Bloch in [5] and which inspired Suslin to define and study the group $B(\mathcal{F})$ in [43]. Both of these groups are described as the kernel of a map from a group that is generated by elements of the form $[x]$ for x in $\mathcal{F}^\flat$, sending each $[x]$ to the class of $(1-x)\otimes x$ in either $\mathcal{F}^*\otimes\mathcal{F}^*$ or a variant like $\tilde{\wedge}^2\mathcal{F}^*$.

However, as Bloch's constriction uses relative K-theory whilst Suslin's uses group homology of $\mathrm{GL}_2(\mathcal{F})$, there is *a priori* no obvious relation between the groups, nor any obvious way to construct a map between them.

It seems, nevertheless, to be widely expected that these groups should be closely related and our approach now provides the first concrete evidence (in situations in which the groups are non-trivial) to suggest both that these groups should be related in a very natural way, and also that Bloch's group should account for all of the indecomposable K_3-group, at least modulo torsion.

To be specific, if $\mathcal{F}$ is equal to a number field F, then $\psi_{F,\mathrm{tf}}$ is injective by the proof of Theorem 3.29, so by Proposition 3.4 the composite map (3.33) is an injection of a finitely generated free abelian group into itself. One can therefore try to determine the size of its cokernel (or, equivalently, of the cokernel of $\psi_{F,\mathrm{tf}}$) by comparing the result of the regulators on $\mathrm{im}(\psi_{F,\mathrm{tf}})$ and on $K_3(F)^{\mathrm{ind}}_{\mathrm{tf}}$.

This observation combines with extensive evidence that we have obtained by computer calculations in the case that F is imaginary quadratic (cf. §6) to motivate us to formulate the following conjecture.

Conjecture 3.34. *If F is a number field, then $\psi_{F,\mathrm{tf}}$ is an isomorphism.*

Remark 3.35. As mentioned above, the map $\psi_{F,\mathrm{tf}}$ is injective and so the main point of Conjecture 3.34 is that $\mathrm{im}(\psi_{F,\mathrm{tf}}) = K_3(F)^{\mathrm{ind}}_{\mathrm{tf}}$, and hence that the group defined by Bloch in [5] accounts for all of $K_3(F)^{\mathrm{ind}}_{\mathrm{tf}}$. However, for a general number field F, this equality would not itself resolve the problem of finding an explicit description of the resulting composite isomorphism in (3.33). Of course, if F is an imaginary quadratic number field, then all groups occurring in the composite are isomorphic to $\mathbb{Z}$ and so the conjecture would imply that (3.33) is multiplication by ± 1. (Recall that $\psi_{F,\mathrm{tf}}$ is itself natural up to a universal choice of sign since this is true for φ_F.)

It would also seem reasonable to hope that (3.33) has a very simple description for any infinite field $\mathcal{F}$, such as, perhaps, being given by multiplication by some integer that is independent of $\mathcal{F}$. Assuming this to be the case, our numerical calculations would imply that this integer is ± 1. If true, this would in turn imply that the isomorphism $K_3(\mathcal{F})^{\mathrm{ind}}_{\mathrm{tf}} \to B(\mathcal{F})_{\mathrm{tf}}$ constructed by Suslin in [43] could be given a more direct, and more directly K-theoretical, description, at least up to sign, as the inverse of the composite isomorphism $B(\mathcal{F})_{\mathrm{tf}} \to \overline{B}(\mathcal{F})_{\mathrm{tf}} \to K_3(\mathcal{F})_{\mathrm{tf}}^{\mathrm{ind}}$ where the first map is induced by Theorem 3.23 and the second is $\psi_{F,\mathrm{tf}}$.

4. A geometric construction of elements in the modified Bloch group

Let k be an imaginary quadratic number field with ring of algebraic integers $\mathcal{O}$.

In this section, we shall use a geometric construction, the Voronoi theory of Hermitian forms, in order to construct a non-trivial element β_{geo} in $\overline{B}(k) \simeq \mathbb{Z}$.

To do this we shall invoke a tessellation of hyperbolic 3-space for k, based on perfect forms, in order to construct an element of the kernel of the homomorphism $\mathrm{d}\colon C_3(\mathcal{L}) \to C_2(\mathcal{L})$ that occurs in (3.11) with $F = k$ and $\nu' = \{1\}$. By applying $f_{3,k}$ to this element we shall then obtain β_{geo} by using the commutativity of the diagram (3.16).

Furthermore, we are able to explicitly determine the image of this element under the regulator map and compare it to the special value $\zeta_k'(-1)$ by using a celebrated formula of Humbert. This will in particular show that the element $\psi_k(\beta_{\mathrm{geo}})$ of $K_3(k)_{\mathrm{tf}}^{\mathrm{ind}}$ that is constructed in this geometric fashion generates a subgroup of index $|K_2(\mathcal{O})|$ (cf. Corollary 4.10(i)).

4.1. **Voronoi theory of Hermitian forms.** Our main tool is the polyhedral reduction theory for $\mathrm{GL}_2(\mathcal{O})$ developed by Ash [1, Chap. II] and Koecher [31], generalizing work of Voronoi [45] on polyhedral reduction domains arising from the theory of perfect forms. See [51, §3] as well as [18, §2 and §6] for details and a description of the algorithms involved. We recall some of the details here to set notation.

We fix a complex embedding $k \hookrightarrow \mathbb{C}$ and identify k with its image. We extend this identification to vectors and matrices as well. We use $\bar{\cdot}$ to denote complex conjugation on $\mathbb{C}$, the non-trivial Galois automorphism on k. Let $V = \mathcal{H}^2(\mathbb{C})$ be the 4-dimensional real vector space of 2×2 complex Hermitian matrices with complex coefficients. Let $C \subset V$ denote the codimension 0 open cone of positive definite matrices. Using the chosen complex embedding of k, we can view $\mathcal{H}^2(k)$, the 2×2 Hermitian matrices with coefficients in k, as a subset of V. Define a map $q\colon \mathcal{O}^2 \setminus \{0\} \to \mathcal{H}^2(k)$ by $q(x) = x\overline{x}^t$. For each $x \in \mathcal{O}^2$, we have that $q(x)$ is on the boundary of C. Let C^* denote the union of C and the image of q.

The group $\mathrm{GL}_2(\mathbb{C})$ acts on V by $g \cdot A = gA\overline{g}^t$. The image of C in the quotient of V by positive homotheties can be identified with hyperbolic 3-space $\mathbb{H}$. The image of q in this quotient is identified with $\mathbb{P}^1_k$, the set of cusps. The action induces an action of $\mathrm{GL}_2(k)$ on $\mathbb{H}$ and the cusps of $\mathbb{H}$ that is compatible with other models of $\mathbb{H}$ (see [20, Chap. 1] for descriptions of other models). We let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1_k$.

Each $A \in V$ defines a Hermitian form $A[x] = \overline{x}^t Ax$, for $x \in \mathbb{C}^2$. Using the chosen complex embedding of k, we can view $\mathcal{O}^2$ as a subset of $\mathbb{C}^2$.

Definition 4.1. For $A \in C$, we define the *minimum of A* as

$$\min(A) := \inf_{x \in \mathcal{O}^2 \setminus \{0\}} A[x].$$

Note that $\min(A) > 0$ since A is positive definite. A vector $v \in \mathcal{O}^2$ is called a *minimal vector of A* if $A[v] = \min(A)$. We let $\mathrm{Min}(A)$ denote the set of minimal vectors of A.

These notions depend on the fixed choice of the imaginary quadratic field k. Since $\mathcal{O}^2$ is discrete in the topology of $\mathbb{C}^2$, a compact set $\{z \mid A[z] \leq \text{bound}\}$ in $\mathbb{C}^2$ gives a finite set in $\mathcal{O}^2$. Thus $\mathrm{Min}(A)$ is finite.

Definition 4.2. We say a Hermitian form $A \in C$ is a *perfect Hermitian form over* k if

$$\mathrm{span}_{\mathbb{R}}\{q(v) \mid v \in \mathrm{Min}(A)\} = V.$$

By a *polyhedral cone* in V we mean a subset σ of the form

$$\sigma = \left\{ \sum_{i=1}^{n} \lambda_i q(v_i) \mid \lambda_i \geq 0 \right\},$$

where $v_1, \ldots, v_n$ are non-zero vectors in $\mathcal{O}^2$. A set of polyhedral cones S forms a *fan* if the following two conditions hold. Note that a face here can be of codimension higher than 1.

(1) If σ is in S and τ is a face of σ, then τ is in S.
(2) If σ and σ' are in S, then $\sigma \cap \sigma'$ is a common face of σ and σ'.

The reduction theory of Koecher [31] applied in this setting gives the following theorem.

Theorem 4.3. *There is a fan* $\tilde{\Sigma}$ *in* V *with* $\mathrm{GL}_2(\mathcal{O})$*-action such that the following hold.*

(i) There are only finitely many $\mathrm{GL}_2(\mathcal{O})$*-orbits in* $\tilde{\Sigma}$.
(ii) Every $y \in C$ *is contained in the interior of a unique cone in* $\tilde{\Sigma}$.
(iii) Any cone $\sigma \in \tilde{\Sigma}$ *with non-trivial intersection with* C *has finite stabiliser in* $\mathrm{GL}_2(\mathcal{O})$.
(iv) The 4*-dimensional cones in* $\tilde{\Sigma}$ *are in bijection with the perfect forms over* k.

The bijection in claim (iv) of this result is explicit and allows one to compute the structure of $\tilde{\Sigma}$ by using a modification of Voronoi's algorithm [18, §2, §6]. Specifically, σ is a 4-dimensional cone in $\tilde{\Sigma}$ if and only if there exists a perfect Hermitian form A such that

$$\sigma = \left\{ \sum_{v \in \mathrm{Min}(A)} \lambda_v q(v) \;\middle|\; \lambda_v \geq 0 \right\}.$$

Modulo positive homotheties, the fan $\tilde{\Sigma}$ descends to a $\mathrm{GL}_2(\mathcal{O})$-tessellation of $\mathbb{H}$ by ideal polytopes. The output of the computation described above is a collection of finite sets Σ_n^*, $n = 1, 2, 3$, of representatives of the $\mathrm{GL}_2(\mathcal{O})$-orbits of the n-dimensional cells in $\mathbb{H}^*$ that meet $\mathbb{H}$. Let $\Sigma^* = \Sigma_1^* \cup \Sigma_2^* \cup \Sigma_3^*$. The cells in Σ^* each have vertices described explicitly by finite sets of vectors in $\mathcal{O}^2$.

4.2. **Bloch elements from ideal tessellations of hyperbolic space.** The collection of 3-dimensional cells

$$\Sigma_3^* = \{P_1, P_2, \ldots, P_m\}$$

above gives rise, after choosing a triangulation of each, to an element in $\overline{B}(k)$, as follows.

We first establish a useful interpretation of a classical formula of Humbert in this setting. For the sake of brevity, we shall write Γ for $\mathrm{PGL}_2(\mathcal{O})$.

Lemma 4.4. *Let* Γ_{P_i} *denote the stabiliser in* Γ *of* P_i*. Then one has*

$$\sum_{i=1}^{m} \frac{1}{|\Gamma_{P_i}|} \mathrm{vol}(P_i) = -\pi \cdot \zeta_k'(-1)\,.$$

Proof. One has

$$\sum_{i=1}^{m} \frac{1}{|\Gamma_{P_i}|} \operatorname{vol}(P_i) = \operatorname{vol}(\mathrm{PGL}_2(\mathcal{O})\backslash\mathbb{H}) = \frac{1}{8\pi^2}|D_k|^{\frac{3}{2}} \cdot \zeta_k(2). \tag{4.5}$$

Here the first equality is clear and the second is a celebrated result of Humbert (see [8], where the formula is given for general number fields).

The claimed formula now follows since an analysis of the functional equation (2.13) shows that the final term in (4.5) is equal to $-\pi \cdot \zeta_k'(-1)$. □

We next subdivide each polytope P_i into ideal tetrahedra $T_{i,j}$ with positive volume without introducing any new vertices,

$$P_i = T_{i,1} \cup T_{i,2} \cup \ldots \cup T_{i,n_i}\,. \tag{4.6}$$

Here we assume that the subdivision is such that the faces of the tetrahedra that lie in the interior of the P_i match. An ideal tetrahedron T with vertices v_1, v_2, v_3, v_4 has volume

$$\operatorname{vol}(T) = D(\mathrm{cr}_3(v_1, v_2, v_3, v_4)).$$

Here D denotes the Bloch-Wigner dilogarithm defined in §3.1.2 above and cr_3 denotes the cross-ratio discussed in §3.3.1. The ordering of vertices is chosen so that the right hand side is positive.

To ease the notation, we let $r_{i,j}$ denote a resulting cross-ratio for $T_{i,j}$. We note that, whilst there is some ambiguity in choosing the order the four vertices of $T_{i,j}$ when defining this cross-ratio, the transformation rules in Remark 3.13 combine with the relations in (3.19) to imply that the induced element $[r_{i,j}]$ of $\overline{\mathfrak{p}}(k)$ is indeed independent of that choice.

We can now formulate the main result of this section (the proof of which will be given in §5).

By Corollary A.5 we know that each $|\Gamma_{P_i}|$ divides 24, so the coefficients in the next theorem are integers. We also note that, by Proposition 3.31, the map $\overline{\mathfrak{p}}(\mathbb{Q}) \to \overline{\mathfrak{p}}(k)$ is injective unless $k = \mathbb{Q}(\sqrt{-1})$, that the map $2 \cdot \overline{\mathfrak{p}}(\mathbb{Q}) \to \overline{\mathfrak{p}}(k)$ is always injective, and that $2 \cdot \overline{\mathfrak{p}}(\mathbb{Q})$ is torsion-free. Moreover, the composition of the map $\overline{\mathfrak{p}}(\mathbb{Q}) \to \overline{\mathfrak{p}}(k)$ with $\partial_{2,k}\colon \overline{\mathfrak{p}}(k) \to \tilde{\wedge}^2 k^*$ is injective if $k \neq \mathbb{Q}(\sqrt{-1})$, and if $k = \mathbb{Q}(\sqrt{-1})$ then this composition has the same kernel as the map $\overline{\mathfrak{p}}(\mathbb{Q}) \to \overline{\mathfrak{p}}(k)$. Therefore the image of $\overline{\mathfrak{p}}(\mathbb{Q})$ in $\overline{\mathfrak{p}}(k)$ always injects into $\tilde{\wedge}^2 k^*$ under $\partial_{2,k}$.

Theorem 4.7. *Let k be an imaginary quadratic number field, with the polytopes P_i and cross-ratios $[r_{i,j}]$ chosen as above. Then the following claims are valid.*

(i) There exists a unique element $\beta_{\mathbb{Q}}$ in the image of $\overline{\mathfrak{p}}(\mathbb{Q})$ in $\overline{\mathfrak{p}}(k)$, such that the element

$$\beta_{\mathrm{geo}} = \beta_{\mathbb{Q}} + \sum_{i=1}^{m} \frac{24}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} [r_{i,j}]$$

belongs to $\overline{B}(k)$. If $k \neq \mathbb{Q}(\sqrt{-2})$ then $\beta_{\mathbb{Q}}$ belongs to the image of $2 \cdot \overline{\mathfrak{p}}(\mathbb{Q})$. In all cases the element β_{geo} is independent of the choice of representatives in Σ_3^*, and the resulting subdivision (4.6) into tetrahedra.

(ii) If no stabiliser of an element in Σ_3^* or Σ_2^* has order divisible by 4, then there is a unique $\tilde{\beta}_{\mathbb{Q}}$ in $\overline{\mathfrak{p}}(\mathbb{Q})$, which lies in $2 \cdot \overline{\mathfrak{p}}(\mathbb{Q})$, such that the element

$$\tilde{\beta}_{\mathrm{geo}} = \tilde{\beta}_{\mathbb{Q}} + \sum_{i=1}^{m} \frac{12}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} [r_{i,j}]$$

belongs to $\overline{B}(k)$*. Moreover, one has* $2\cdot\tilde{\beta}_{\rm geo} = \beta_{\rm geo}$ *and* $2\cdot\tilde{\beta}_{\mathbb{Q}} = \beta_{\mathbb{Q}}$*.*

Remark 4.8. The situation for k equal to either $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$ is more complicated because the order of the stabiliser of the (in both cases unique) element of Σ_3^* has order 24. For $k=\mathbb{Q}(\sqrt{-2})$ it can be subdivided in several different ways, resulting in the exception in Theorem 4.7(i). In fact, the subdivision in this case determines whether $\beta_{\mathbb{Q}}$ either belongs, or does not belong, to $2\cdot\overline{\mathfrak{p}}(\mathbb{Q})$, and both cases occur; see the argument in §5 below for more details.

Remark 4.9. It is sometimes computationally convenient to avoid explicitly computing the element $\beta_{\mathbb{Q}}$ in Theorem 4.7(i). In this regard it is useful to note that the injectivity in Corollary 3.30 combines with Theorem 3.4(iii) and the equality $D(\overline{z}) = -D(z)$ in (3.3) to imply that

$$2\cdot\beta_{\rm geo} = \sum_{i=1}^{m}\frac{24}{|\Gamma_{P_i}|}\sum_{j=1}^{n_i}([r_{i,j}] - [\overline{r_{i,j}}]).$$

4.3. **Regulator maps and K-theory.** As we fixed an injection of k into $\mathbb{C}$, by the behaviour of the regulator map reg_2 with respect to complex conjugation (see (3.3)), we can compute regulators by considering only the composition

$$K_3(k)\to K_3(\mathbb{C})\xrightarrow{\mathrm{reg}_2}\mathbb{R}(1).$$

By slight abuse of notation, we shall denote this composition by reg_2 as well.

Corollary 4.10. *Assume the notation and hypotheses of Theorem 4.7. Then the following claims are valid.*

(i) The element $\psi_k(\beta_{\rm geo})$ *satisfies*

$$\frac{\mathrm{reg}_2(\psi_k(\beta_{\rm geo}))}{2\pi i} = -12\cdot\zeta_k'(-1)$$

and generates a subgroup of the infinite cyclic group $K_3(k)^{\rm ind}_{\rm tf}$ *of index* $|K_2(\mathcal{O})|$*.*

(ii) If no stabiliser of an element in Σ_3^* *or* Σ_2^* *has order divisible by* 4*, then* $|K_2(\mathcal{O})|$ *is even, and* $\psi_k(\tilde{\beta}_{\rm geo})$ *generates a subgroup of* $K_3(k)^{\rm ind}_{\rm tf}$ *of index* $|K_2(\mathcal{O})|/2$*.*

Proof. Before proving claim (i) we note that for each polytope P_i in Σ_3^* one has

$$\sqrt{-1}\cdot\mathrm{vol}(P_i) = \sum_{j=1}^{n_i}\mathbb{D}([r_{i,j}]), \tag{4.11}$$

where $\mathbb{D}$ is the homomorphism $\overline{\mathfrak{p}}(k)\to\mathbb{R}(1)$ that is defined in Remark 3.21 with respect to a fixed embedding $k\to\mathbb{C}$. This is true because $\mathrm{vol}(P_i)=\sum_{j=1}^{n_i}\mathrm{vol}(T_{i,j})$ whilst for each i and j one has $\sqrt{-1}\cdot\mathrm{vol}(T_{i,j}) = \sqrt{-1}\cdot D(r_{i,j}) = \mathbb{D}([r_{i,j}])$.

Turning now to the proof of claim (i), we observe that, since the element $\beta_{\mathbb{Q}}$ that occurs in the definition of $\beta_{\rm geo}$ lies in the image of the map $\overline{\mathfrak{p}}(\mathbb{Q})\to\overline{\mathfrak{p}}(k)$, it also lies in the kernel of the composite homomorphism $\mathrm{reg}_2 \circ \psi_k$. One therefore computes that

$$\begin{aligned} \mathrm{reg}_2(\psi_k(\beta_{\mathrm{geo}})) &= \sum_{i=1}^{m} \frac{24}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} \mathrm{reg}_2(\psi_k([r_{i,j}])) \\ &= 24 \cdot \sum_{i=1}^{m} \frac{1}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} \mathbb{D}([r_{i,j}]) \\ &= 24\sqrt{-1} \cdot \sum_{i=1}^{m} \frac{1}{|\Gamma_{P_i}|} \mathrm{vol}(P_i) \\ &= -24\pi\sqrt{-1} \cdot \zeta_k'(-1) \end{aligned}$$

,

where the second equality follows from Theorem 3.4(iii) and Remark 3.21, the third from (4.11) and the last from Lemma 4.4.

The first assertion of claim (i) follows immediately from this displayed equality and, given this, the final assertion of claim (i) follows directly from the equality (2.12).

For claim (ii) we note that, under the stated conditions, Theorem 4.7(ii) implies $\beta_{\mathrm{geo}} = 2 \cdot \tilde{\beta}_{\mathrm{geo}}$, so this follows from the final assertion of claim (i). □

Remark 4.12.
(i) The results of Theorem 4.7(ii) and Corollary 4.10(ii) apply to many fields $\mathbb{Q}(\sqrt{-d})$, the first few when ordered by d being $\mathbb{Q}(\sqrt{-15})$, $\mathbb{Q}(\sqrt{-30})$, $\mathbb{Q}(\sqrt{-35})$, $\mathbb{Q}(\sqrt{-39})$, and $\mathbb{Q}(\sqrt{-42})$. In the first, third and fifth case here one has $|K_2(\mathcal{O})| = 2$ and so $\psi_k(\tilde{\beta}_{\mathrm{geo}})$ generates $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$.

(ii) The example discussed in §5.3 below shows that one cannot ignore the condition on the stabilisers of the elements of Σ_2^* in Theorem 4.7(ii) Corollary 4.10(ii). Specifically, in this case the stabilisers of the elements of Σ_3^* have order 2 or 3, one element of Σ_2^* has stabiliser of order 4, but β_{geo} generates $\overline{B}(k)$ so cannot be divided by 2.

4.4. **A cyclotomic description of** β_{geo}**.** Let k be an imaginary quadratic field of conductor N and set $\zeta_N := e^{2\pi i/N}$. Then we may fix an injection of k into the cyclotomic subfield $F = \mathbb{Q}(\zeta_N)$ of $\mathbb{C}$.

The following result shows that the image under the induced map $\overline{B}(k) \longrightarrow \overline{B}(F)$ of the element β_{geo} constructed in Theorem 4.7(i) has a simple description in terms of elements constructed directly from roots of unity. (This result is, however, of very limited practical use since it is generally much more difficult to compute explicitly in $\overline{\mathfrak{p}}(F)$ rather than in $\overline{\mathfrak{p}}(k)$).

Proposition 4.13. *The image of* β_{geo} *in* $\overline{B}(F)$ *is equal to* $N \sum_{\overline{a} \in \mathrm{Gal}(F/k)} [\zeta_N^a]$.

Proof. At the outset we note that there is a commutative diagram

$$\begin{array}{ccccccc} K_3(k)^{\mathrm{ind}}_{\mathrm{tf}} & \xrightarrow{\simeq} & \overline{B}(k) & \xrightarrow{\psi_k} & K_3(k)^{\mathrm{ind}}_{\mathrm{tf}} & \xrightarrow{\mathrm{reg}_2} & \mathbb{R}(1) \\ \downarrow & & \downarrow & & \downarrow & & \| \\ K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} & \xrightarrow{\simeq} & \overline{B}(F) & \xrightarrow{\psi_F} & K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} & \xrightarrow{\mathrm{reg}_2} & \mathbb{R}(1)\,, \end{array}$$

where the isomorphisms are obtained from (3.22), as well as Theorems 3.23 and 3.29, and reg_2 is the regulator map corresponding to our chosen embeddings of k and F into $\mathbb{C}$.

We further recall (from, for example, [42, Prop. 5.13] with $X = Y' = \mathrm{Spec}(F)$ and $Y = \mathrm{Spec}(k)$) that the composition $K_3(F) \to K_3(k) \to K_3(F)$ of the norm and pullback is given by the trace, and that the same is also true for the induced maps on $K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} = K_3(F)_{\mathrm{tf}}$ and $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}} = K_3(k)_{\mathrm{tf}}$. By applying this fact to the element of $K_3(F)^{\mathrm{ind}}_{\mathrm{tf}}$ corresponding to $N[\zeta_N]$ in $\overline{B}(F)$, we deduce from the left hand square in the above diagram that there exists an element β_{cyc} in $\overline{B}(k)$ that maps to $N\sum_{\overline{a}\in\mathrm{Gal}(F/k)}[\zeta_N^a]$ in $\overline{B}(F)$.

We now identify $\mathrm{Gal}(F/k)$ with a subgroup of index 2 of $(\mathbb{Z}/N\mathbb{Z})^*$, which is the kernel of a primitive character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ of order 2, corresponding to k (so $\chi(\overline{-1}) = -1$). Then from the above diagram and Theorem 3.4(iii), one finds that $(2\pi i)^{-1} \cdot \mathrm{reg}_2(\psi_k(\beta_{\mathrm{cyc}}))$ is equal to

$$\begin{aligned}\frac{N}{4\pi i}\sum_{\overline{a}\in\mathrm{Gal}(F/k)}\sum_{n\geq 1}\frac{\zeta_N^{na}-\zeta_N^{-na}}{n^2} &= \frac{N}{4\pi i}\sum_{\overline{a}\in\mathrm{Gal}(F/\mathbb{Q})}\sum_{n\geq 1}\chi(\overline{a})\frac{\zeta_N^{na}}{n^2}\\ &= \frac{N^{3/2}}{4\pi}L(\mathbb{Q},\chi,2)\\ &= -12\zeta_k'(-1)\end{aligned}$$

as $\zeta_k(s) = \zeta_{\mathbb{Q}}(s)L(\mathbb{Q},\chi,s)$, with the Gauß sum $\sum_{\overline{a}\in\mathrm{Gal}(F/\mathbb{Q})}\chi(\overline{a})\zeta_N^a = i\sqrt{N}$ (see [25, §58]). (Cf. the more general (and involved) calculation of [52, p. 421], or the calculation in the proof of [16, Th. 3.1] with $r = -1$, $\ell = 1$ and $\mathcal{O} = \mathbb{Z}$.)

According to Theorem 4.7, one has $(2\pi i)^{-1} \cdot \mathrm{reg}_2(\psi_k(\beta_{\mathrm{geo}})) = -12 \cdot \zeta_k'(-1)$ and so, by the injectivity of reg_2 on $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$ (cf. Corollary 3.30) one has $\psi_k(\beta_{\mathrm{geo}}) = \psi_k(\beta_{\mathrm{cyc}})$. It then follows that $\beta_{\mathrm{geo}} = \beta_{\mathrm{cyc}}$ because ψ_k is injective. □

5. The proof of Theorem 4.7

Throughout this section we fix an imaginary quadratic field k with ring of integers $\mathcal{O}$, as in §4. In §5.2 we also use the embedding of k into $\mathbb{C}$ chosen there.

5.1. **A preliminary result concerning orbits.** We start by proving a technical result that will play an important role in later arguments.

We set $V = k^2 \setminus \{(0,0)\}$ and let Γ denote either $\mathrm{SL}_2(\mathcal{O})$ or $\mathrm{GL}_2(\mathcal{O})$.

Lemma 5.1.

(i) For v in V, $\mathcal{O}^*$ acts on the orbit Γv, and the natural map $V \to \mathbb{P}^1_k$ induces an injection of $\Gamma v/\mathcal{O}^*$ into $\mathbb{P}^1_k$, compatible with the action of Γ.

(ii) For v_1 and v_2 in V, the images of $\Gamma v_1/\mathcal{O}^*$ and $\Gamma v_2/\mathcal{O}^*$ are either disjoint or coincide.

Proof. That $\mathcal{O}^*$ acts on Γv is clear if $k \neq \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ as $\mathcal{O}^* = \{\pm 1\}$ and Γ contains $\pm\mathrm{id}_2$. For the two remaining cases, $\mathcal{O}$ is Euclidean, and based on iterated division with remainder in $\mathcal{O}$ it is easy to find g in $\mathrm{SL}_2(\mathcal{O})$ with $gv = \left(\begin{smallmatrix} c \\ 0 \end{smallmatrix}\right)$ for some c in k^*, so if u is in $\mathcal{O}^*$, then uv is in the orbit of v because $g^{-1}\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)g$ maps v to uv. Alternatively, this follows immediately from [20, Chap. 7, Lem. 2.1] because if $v = (\alpha, \beta)$, and u is in $\mathcal{O}^*$, then $(\alpha,\beta) = (u\alpha, u\beta)$.

Now assume that $g_1 v = c g_2 v$ with c in k^* and the g_i in Γ. Then v is an eigenvector with eigenvalue c of the element $g_2^{-1}g_1$ in Γ, which has determinant in $\mathcal{O}^*$. Hence c is in $\mathcal{O}^*$, and $g_1 v$

and $g_2 v$ give the same element in $\Gamma v/\mathcal{O}^*$. So we do get the claimed injection, and it is clearly compatible with the action of Γ.

For the last part, suppose that $g_1 v_1 = c g_2 v_2$ for some c in k^*, v_i in V, and g_i in Γ. Then $\Gamma v_1 = c\Gamma v_2$ and the result is clear. □

Proposition 5.2. *If h is the class number of k, then we can find $v_1, \ldots, v_h$ in V such that $\mathbb{P}^1_k$ is the disjoint union of the images of the $\Gamma v_i/\mathcal{O}^*$. In particular, every element in $\mathbb{P}^1_k$ lifts uniquely to some $\Gamma v_i/\mathcal{O}^*$ and this lifting is compatible with the action of Γ.*

Proof. By [20, Chap. 7, Lem. 2.1] we may identify $\Gamma\backslash V$ with the set of fractional ideals of $\mathcal{O}$ and hence $\Gamma\backslash V/k^* = \Gamma\backslash\mathbb{P}^1_k$ with the ideal class group of k. Given this, the claimed result follows directly from Lemma 5.1. □

5.2. **The proof of Theorem 4.7.**

5.2.1. We first establish some convenient notation and conventions.

For a 2-cell, or more generally, any flat polytope with vertices $v_1, \ldots, v_n$ in that order along its boundary, we indicate an orientation by $[v_1, \ldots, v_n]$ up to cyclic rotation. The inverse orientation corresponds to reversing the order of the vertices. If we want to denote the face with either orientation, we write $(v_1, \ldots, v_n)$. In particular, an orientated triangle is the same as a 3-tuple $[v_1, v_2, v_3]$ of vertices up to the action (with sign) of S_3. Similarly, an orientated tetrahedron is the same as a 4-tuple $[v_1, v_2, v_3, v_4]$ up to the action (with sign) of S_4. Recall that we defined maps $f_{3,k}$ and $f_{2,k}$ just after (3.12). As mentioned above, the map $f_{3,k}$ is compatible with the action of S_4 by Remark 3.13 and (3.19). By the properties of cr_2 mentioned just before (3.12), the map $f_{2,k}$ is also compatible with the action of S_3 on orientated triangles if we lift them to elements of $C_3(\mathcal{L})$ for some suitable $\mathcal{L}$.

5.2.2. By our discussion before the statement of the theorem, the uniqueness of elements $\beta_{\mathbb{Q}}$ and $\tilde{\beta}_{\mathbb{Q}}$ with the stated properties is clear. It is also clear that for any element $\beta_{\mathbb{Q}}$ in $\overline{\mathfrak{p}}(k)$ the explicit sum $\tilde{\beta}_{\mathrm{geo}}$ belongs to $\mathfrak{p}k$. In addition, the uniqueness of $\beta_{\mathbb{Q}}$ combines with the explicit expression for $\tilde{\beta}_{\mathrm{geo}}$ to imply that $2\tilde{\beta}_{\mathbb{Q}} = \beta_{\mathbb{Q}}$, hence that $2\tilde{\beta}_{\mathrm{geo}} = \beta_{\mathrm{geo}}$.

The fact that β_{geo} is independent of the subdivision (4.6), and of the choice of representatives in Σ_3^* also follows directly from the equality in Corollary 4.10(i) and the injectivity assertions in Corollary 3.30, once it is known that β_{geo} is in $\overline{B}(k)$.

To prove Theorem 4.7 it is therefore sufficient to prove the existence of elements $\beta_{\mathbb{Q}}$ and $\tilde{\beta}_{\mathbb{Q}}$ in the stated groups such that the sums β_{geo} and $\tilde{\beta}_{\mathrm{geo}}$ belong to $\overline{B}(k)$ and to do this we shall use the tessellation.

This argument is given in the next subsection. The basic idea is that, for k different from $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$, the sum $12 \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}]$ has integer coefficients and belongs to the kernel of $\partial_{2,k}$ since the faces of the polytopes P_i with those multiplicities can be matched under the action of Γ. This argument uses that $f_{2,k}$ is invariant under the action of $\mathrm{GL}_2(k)$ and behaves compatibly with respect to permutations, just as $f_{3,k}$.

The precise argument is slightly more complicated because the subdivision (4.6) induces triangulations of the faces of the P_i which may not correspond, necessitating the introduction of 'flat tetrahedra', which give rise to the term $\beta_{\mathbb{Q}}$. Also, the faces themselves may have orientation reversing elements in their stabilisers. But the resulting matching of faces does imply that the explicit sum β_{geo} lies in the kernel $\overline{B}(k)$ of $\partial_{2,k}$.

For the special cases $k = \mathbb{Q}(\sqrt{-1})$, $k = \mathbb{Q}(\sqrt{-2})$, and $k = \mathbb{Q}(\sqrt{-3})$, we have to compute more explicitly for the single polytope involved in each case.

5.2.3. We note first that each polytope P in the tessellation of $\mathbb{H}$ comes with an orientation corresponding to it having positive volume. For a face (2-cell) F in the tessellation, we fix an orientation, and consider the group $\oplus_F \mathbb{Z}[F]$, where we identify $[F^\dagger]$ with $-[F]$ if $F^\dagger$ denotes F with the opposite orientation.

To any P we associate its boundary ∂P in this group, where each face has the induced orientation. As the action of Γ on $\mathbb{H}$ preserves the orientation, it commutes with this boundary map.

We now need to do some counting. For a face $[F]$, we let Γ_F denote the stabiliser of the (non-oriented) face F, and Γ_F^+ the subgroup that preserves the orientation $[F]$. We note that the index of Γ_F^+ in Γ_F is either 1 or 2.

Let P and P' be the polytopes in the tessellation that have F in their boundaries. If g is in Γ_F then $gP = P$ or P', and $gP = P$ precisely when g is in Γ_F^+. Therefore $\Gamma_F^+ = \Gamma_F \cap \Gamma_P$.

It is convenient to distinguish between the following two cases for the Γ-orbits of F.

- $\Gamma_F = \Gamma_F^+$. If P and P' in Σ_3^* are such that their boundaries each contain an element in the Γ-orbit of $[F]$, then P and P' are in the same Γ-orbit, hence are the same. Therefore there is exactly one P in Σ_3^* that contains faces in the Γ-orbit of $[F]$. If two faces of P are in the Γ-orbit of $[F]$, then they are transformed into each other already by Γ_P. Hence the number of elements in the Γ-orbit of F in ∂P is $[\Gamma_P : \Gamma_F] = [\Gamma_P : \Gamma_F^+]$. If P' is the element in Σ_3^* that has an element in the Γ-orbit of $[F^\dagger]$ in its boundary (with $P = P'$ and $P \neq P'$ both possible), then there are $[\Gamma_{P'} : \Gamma_F] = [\Gamma_{P'} : \Gamma_F^+]$ elements in the Γ-orbit of $[F^\dagger]$ in the boundary of P'.
- $\Gamma_F \neq \Gamma_F^+$. Note that in this case $[\Gamma_F : \Gamma_F^+] = 2$. Here $[F]$ and $[F^\dagger]$ are in the same Γ-orbit and as above one sees that there is only one element P of Σ_3^* that has elements in this Γ-orbit in its boundary. Any two such elements can be transformed into each other using elements of Γ_P, so there are $[\Gamma_P : \Gamma_F^+]$ of those in the boundary of P.

5.2.4. In this subsection we prove Theorem 4.7 in the general case that k is neither $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, nor $\mathbb{Q}(\sqrt{-3})$.

In this case Corollary A.5 implies that the order of each group Γ_{P_i} divides 12, and so both the formal sum of elements in Σ_3^* given by

$$\pi_P = \sum_{i=1}^{m} \frac{12}{|\Gamma_{P_i}|}[P_i]\,,$$

and the formal sum of of tetrahedra resulting from the subdivision (4.6)

$$\pi_T = \sum_{i=1}^{m} \frac{12}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} [T_{i,j}]\,,$$

have integral coefficients.

We extend the boundary map ∂ to such formal sums, where for π_T the boundary is a formal sum of ideal triangles, contained in the original faces of the polytopes P_i. (Note that the subdivision (4.6) may introduce 'internal faces' inside each polytopes, but by construction the parts of the boundaries of the tetrahedra here cancel exactly. This also holds after lifting all vertices to $\mathcal{O}^2$ as there is no group action involved in order to match them. So we may, and shall, ignore those internal faces.)

The subdivision (4.6) induces a triangulation of all the faces of each P_i in Σ_3^*. We let $[\Delta_F]$ denote the induced triangulation. But if F is a face of such a P_i with $[F] \neq [F^\dagger]$, then the induced triangulations $[\Delta_F]$ and $[\Delta_{F^\dagger}]$ (which may come from a different element in Σ_3^*) may not match. Similarly, if gF and F are both faces of P_i with g in Γ_{P_i}, then $g[\Delta_F]$ and $[\Delta_{gF}]$ may not match.

A typical example of non-matching triangulations is that where F is a 'square' $[v_1, v_2, v_3, v_4]$ that is cut into two triangles using either diagonal, resulting in the triangulations $[v_1, v_2, v_4] + [v_2, v_3, v_4]$ and $[v_1, v_2, v_3]+[v_1, v_3, v_4]$. But the boundary of the orientated tetrahedron $[v_1, v_2, v_3, v_4]$ gives exactly the former minus the latter. It is easy to see (e.g., by induction on the number of vertices of F) that by using boundaries of such tetrahedra we can change any triangulation of $[F]$ into any other with the same orientation. Because the tetrahedra are contained in the faces, they have no volume, and the cross ratios of the four cusps is in $\mathbb{Q}^\flat$. We refer to them as 'flat tetrahedra', and if $[\Delta_F^1]$ and $[\Delta_F^2]$ are two triangulations (with the same orientation) of an orientated face $[F]$, we shall write

$$\text{`}[\Delta_F^1] \equiv [\Delta_F^2] \text{ mod. } \partial(\text{flat tetrahedra})\text{'}$$

if $[\Delta_F^1] - [\Delta_F^2]$ is the boundary of a formal sum of such flat tetrahedra.

In particular, if $[\Delta_F]$ and $[\Delta_{F^\dagger}]$ are any triangulations of the faces $[F]$ and $[F^\dagger]$ (so with opposite orientation), then $[\Delta_F] + [\Delta_{F^\dagger}] \equiv 0$ mod. ∂(flat tetrahedra).

We extend the boundary map to the free abelian group $\oplus_{P\in\Sigma_3^*}\mathbb{Z}[P]$ by linearity. For a given $[F]$, in $\partial(\pi_P)$ we find find $12 \cdot |\Gamma_F^+|^{-1}$ copies of $[F]$ (up to the action of Γ). If $[F] \neq [F^\dagger]$ (i.e., if $\Gamma_F = \Gamma_F^+$) then we have the same number of copies of $[F^\dagger]$, and we deal with $[F]$ and $[F^\dagger]$ together.

We now distinguish four cases in terms of the number of factors of 2 in $|\Gamma_F|$ and $|\Gamma_F^+|$. Note that Γ_F^+ is cyclic, so by Lemma A.2 and our assumptions on k we can write its order as $2^s m$ with $m = 1$ or 3, and $s = 0$ or 1, with the case $m = 3$ and $s = 1$ not occuring. Then $|\Gamma_F| = 2^t|\Gamma_F^+|$ with $t = 0$ or 1.

(1) $s = t = 0$. Here $F^\dagger$ is not in the same Γ-orbit as F, and in $\partial(\pi_P)$ the contribution of their Γ-orbits is $\frac{12}{m}[F] + \frac{12}{m}[F^\dagger]$, modulo the action of Γ. Then for $\partial(\pi_T)$ they contribute $\frac{12}{m}[\Delta_F] + \frac{12}{m}[\Delta_F^\dagger]$ mod. ∂(flat tetrahedra) and modulo the action of Γ.
(2) $s = 0$ and $t = 1$. Here F and $F^\dagger$ are in the same Γ-orbit, and in the boundary of π_P, up to the action of Γ, we have $\frac{12}{m}[F] = \frac{6}{m}[F]+\frac{6}{m}[F^\dagger]$. Then in $\partial(\pi_T)$ we obtain $\frac{6}{m}[\Delta_F]+\frac{6}{m}[\Delta_F^\dagger]$ mod. ∂(flat tetrahedra) and modulo the action of Γ.
(3) $s = 1$ and $t = 0$. This is similar to case (1), but now $[F]$ and $[F^\dagger]$ both occur with coefficient 6 in $\partial(\pi_P)$ because $m = 1$. In $\partial(\pi_T)$ we obtain $6[\Delta_F] + 6[\Delta_F^\dagger]$ mod. ∂(flat tetrahedra), and modulo the action of Γ.
(4) $s = t = 1$. This is similar to case (2), but now in $\partial(\pi_P)$ we find $6[F] = 3[F] + 3[F^\dagger]$, again because $m = 1$, hence in $\partial(\pi_T)$ this gives $3[\Delta_F] + 3[\Delta_F^\dagger]$ mod. ∂(flat tetrahedra) and modulo the action of Γ.

We see that there exists some α, a formal sum of flat tetrahedra, such that $\pi_T+\alpha$ has boundary, up to the action of Γ, a formal sum with terms $[t] + [t^\dagger]$ with t an ideal triangle. Lifting all cusps to $\mathcal{L} = \Gamma v_1/\mathcal{O}^* \coprod \ldots \coprod \Gamma v_h/\mathcal{O}^*$ as in Corollary 5.2, and applying $f_{3,k}$ as in (3.16), we see from the Γ-equivariance of $f_{2,k}$, and the fact that this map is alternating, so kills elements of the form $[t] + [t^\dagger]$, that $\sum_{i=1}^m 12 \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}] + \beta'$ is in the kernel of $\partial_{2,k}^{\mathcal{O}^*}$, where β' is the image of α.

Note that β' lies in the image of $\overline{\mathfrak{p}}(\mathbb{Q})$ in $\overline{\mathfrak{p}}(k)$. Multiplying by $|\mathcal{O}^*| = 2$ and setting $\beta_{\mathbb{Q}} := 2\beta'$ we complete the proof of Theorem 4.7(i) in this case.

The proof of Theorem 4.7(ii) is done in the same way, starting with $\sum_{i=1}^m 6 \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}]$ (which has integer coefficients under the stated assumptions). In this case the coefficients in the above cases (1), (2) and (3) are divided by 2, and case (4) is ruled out by the assumptions.

5.2.5. We now consider the special fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{-3})$.

In each of these cases either $|\Gamma_{P_i}|$ does not divide 12 or $|\mathcal{O}^*|$ is larger than 2. However, one also knows that Σ_3^* has only one element and its stabiliser has order 12 or 24 and so the result of Theorem 4.7(ii) does not apply. It is therefore enough to prove Theorem 4.7(i) for these fields.

If $k = \mathbb{Q}(\sqrt{-1})$, then Σ_3^* is an octahedron, with stabiliser isomorphic to S_4. Using that an ideal tetrahedron with positive volume in this octahedron must contain exactly two antipodal points, it is easy to see that the subdivision is unique up to the action of the stabiliser. Hence the resulting element under $f_{3,k}$ is well-defined. Computing it explicitly as $\sum_{j=1}^4 [r_{1,j}] = 4[\omega]$ one finds that it is in the kernel of $\partial_{2,k}$ as $\omega^2 = -1$, so we can simply take $\beta_{\mathbb{Q}} = 0$.

If $k = \mathbb{Q}(\sqrt{-3})$, then the polytope is a tetrahedron, with stabiliser isomorphic to A_4. Computing its image $[r_{1,1}]$ under $f_{3,k}$ explicitly one finds $[\omega]$ with $\omega^2 = \omega - 1$, and $\partial_{2,k}([r_{1,1}]) = \omega \tilde{\wedge} (1-\omega) = (-1) \tilde{\wedge} (-1)$ in $\tilde{\wedge}^2 k^*$, which has order 2 by Remark 3.7. So we can again take $\beta_{\mathbb{Q}} = 0$.

If $k = \mathbb{Q}(\sqrt{-2})$, then the polytope is a rectified cube (i.e., a cuboctahedron), with stabiliser isomorphic to S_4, so it has six 4-gons and eight triangles as faces. By the commutativity of (3.16), for $\nu' = \{1\}$, we can compute $\partial_{2,k}(\sum_j [r_{1,j}])$ by choosing lifts of all vertices involved, and applying $f_{2,k}$ to each of the lifted triangles (with correct orientation) of the induced triangulation of the faces of P_1. (This provides an alternative approach for $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ as well.) Note that any triangulation of the faces occurs for some subdivision: fix a vertex V and use the cones on all the triangles that do not have V as a vertex.

So we must consider all triangulations. Giving the six 4-gons one of the two possible triangulations at random, resulted in $(-1) \tilde{\wedge} (-1) = (-1) \tilde{\wedge} 2 = \partial_{2,k}([2])$ in $\tilde{\wedge}^2 k^*$. The other triangulations we obtain from this one by triangulating one or more of the 4-gons in the other way. For each 4-gon, this adds the image under $\partial_{2,k}$ of the cross-ratio of the flat tetrahedron corresponding to it. As the 4-gons are all equivalent under Γ, computing this for one suffices. The result is $[2]$ again. So by Proposition 3.31 we find that $\partial_{2,k}(\sum_j [r_{1,j}])$ equals either 0 or $\partial_{2,k}([2])$, and both occur. By Corollary 3.30, we must take $\beta_{\mathbb{Q}} = 0$ or $[2]$, and by Proposition 3.31 the latter is not in $2\overline{\mathfrak{p}}(\mathbb{Q})$.

This completes the proof of Theorem 4.7.

5.2.6. We make several observations concerning the above argument.

Remark 5.3. Let k not be equal to $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$. One can try to find a better element than β_{geo} by going through the calculations in the proof of Theorem 4.7 after replacing π_P by an element of the form $\sum_{i=1}^m M \cdot |\Gamma_{P_i}|^{-1} [P_i]$ for some positive integer M that is divisible by the orders of the stabilisers Γ_{P_i}. If we start with M equal to the least common multiple of the orders $|\Gamma_{P_i}|$, then we may have to multiply this element by 2 perhaps twice in the proof in order to ensure that the resulting element in $\overline{\mathfrak{p}}(k)$ belongs to $\overline{B}(k)$:

(1) in order to ensure that the boundary ∂ of the resulting analogue of π_T is trivial up to the action of Γ, which is not automatic if some P_i has a face with reversible orientation under Γ and $M \cdot |\Gamma_{P_i}|^{-1}$ is odd;
(2) in order to ensure that $\sum_{i=1}^{m} M \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}] + \beta'$ is in the kernel of $\partial_{2,k}$ and not just $\partial_{2,k}^{\mathcal{O}^*}$, where M results from (1), and β' (coming from flat tetrahedra) is in $\overline{\mathfrak{p}}(\mathbb{Q})$, which we view as inside $\overline{\mathfrak{p}}(k)$ by Proposition 3.31.

Note that in the second part here we use $|\mathcal{O}^*|= 2$, which excludes $k = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$.

For our k, with $\mathrm{Nm}\colon k \to \mathbb{Q}$ the norm, the Hermitian form $(x, y) \mapsto \mathrm{Nm}(x)+\mathrm{Nm}(y)+\mathrm{Nm}(x-y)$ on $\mathbb{C}^2$ has minimal vectors $\{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}$. By [18, Th. 2.7], this means that the triangle with vertices 0, 1 and ∞ is a 2-cell of the tessellation. The element $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ in Γ of order 3 stabilises this triangle while preserving its orientation. Therefore the 3-cells that share this triangle as faces have stabilisers with orders divisible by 3, and M is divisible by 3.

Also, the elements $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ have order 2, and generate a subgroup of Γ of order 4. The first has as axis of rotation the 1-cell connecting 0 and ∞, so the axis of rotation of the second, which meets this 1-cell, must meet either a 3-cell, or a 2-cell with vertices 0, ∞, and purely imaginary numbers. In the first case we start with M divisible by 6. In the second case, (2) above ensures that M is even since the 2-cell reverses orientation under $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Because in this remark we are also assuming that $k \neq \mathbb{Q}(\sqrt{-2})$, we know from Corollary A.5 that the greatest common divisor of the orders of the Γ_{P_i} divides 12. So this method could lead to an element β_{geo} as in Theorem 4.7(i) but with 24 replaced by either 6, 12, or 24.

Remark 5.4. In our calculations, we find the following for all fields k that differ from $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$:

- the gcd of the orders of the stabilisers Γ_{P_i} is 6 or 12, i.e., 3 does not occur;
- the sum $\sum_{i=1}^{m} 12 \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} ([r_{i,j}] - [\overline{r_{i,j}}])$ belongs to $\overline{B}(k)$, so that by Remark 4.9 and Corollary 3.30(iii), this must be another expression for β_{geo} in $\overline{B}(k)$.

Unfortunately, we have not been able to prove either of these statements in general.

5.3. **An explicit example.** With the same notation as before Theorem 4.7, we consider the element $\beta' = \sum_{i=1}^{m} M \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}]$ of $\overline{\mathfrak{p}}(k)$, where M is the greatest common divisor of the orders $|\Gamma_{P_i}|$. As in Remark 5.3, the proof of Theorem 4.7 shows that there exists a positive divisor e of $2|\mathcal{O}^*|$ such that $\partial_{2,k}(e\beta')$ is in the image of the composition $\overline{\mathfrak{p}}(\mathbb{Q}) \to \tilde{\bigwedge}^2 \mathbb{Q}^* \to \tilde{\bigwedge}^2 k^*$, and it gives a way of computing this out of the data of the tessellation. But there are some choices involved, for example, the pairing up of faces of the P_i under the action of Γ, which may result in an e that is not optimal.

But one can also do this algebraically, by computing β' and determining a (minimal) positive integer e such that $e\partial_{2,k}(\beta')$ is in the image of $\overline{\mathfrak{p}}(\mathbb{Q})$. For this we can use Corollary 3.8 and Remark 3.9: if S is a finite set of finite places of k such that $\tilde{\bigwedge}^2 \mathcal{O}_S^* \subset \tilde{\bigwedge}^2 k^*$ contains $\delta = \partial_{2,k}(\beta)$, and $A = \mathbb{Q}^* \cap \mathcal{O}_S^*$, then one can compute if $e\delta$ is in the image of $\tilde{\bigwedge}^2 A$ or not. If this is the case then one can compute its preimage in $\tilde{\bigwedge}^2 \mathbb{Q}^*$ using Lemma 3.10, algorithmically determine if an element in there yields the trivial element in $K_2(\mathbb{Q})$, and if so, express it in terms of $\partial_{2,\mathbb{Q}}([x])$ with x in $\mathbb{Q}^\flat$.

Note that a different subdivision (4.6) might a priori give rise to a different β' and a different e, but the other choices are irrelevant in this algebraic approach.

For the reader's convenience we illustrate this, and the methods of the proof of Theorem 4.7, in the special case that $k = \mathbb{Q}(\sqrt{-5})$. In particular, in this case we find that both methods give the same element, which generates $\overline{B}(k)$.

The lifts to $\mathcal{O}^2$ (up to scaling by $\mathcal{O}^*$) of the vertices $v_1, \ldots, v_8$ in the two elements P_1 and P_2 of Σ_3^* are the columns of the matrix

$$\begin{pmatrix} \omega+1 & 1 & 2 & 2 & 0 & -\omega & 1 & -\omega+1 \\ -2 & 0 & \omega-1 & \omega & 1 & 2 & -1 & \omega+1 \end{pmatrix}. \tag{5.5}$$

Both polytopes are triangular prisms, which we write as $[a, b, c; A, B, C]$, where $[a, b, c]$ and $[A, B, C]$ are triangles, with A above a, etc. Such a prism can be subdivided into orientated tetrahedra as $[a, A, B, C]-[a, b, B, C]+[a, b, c, C]$, which results in the subdivision of its orientated boundary as

$$[A, B, C] + [a, A, C] - [a, c, C] + [a, b, B] - [a, A, B] + [b, c, C] - [b, B, C] - [a, b, c]\,. \tag{5.6}$$

Here the first and last terms correspond to the triangular faces, and the middle terms are grouped as pairs of triangles in the rectangular faces.

Then $P_1 = [v_3, v_5, v_4; v_1, v_2, v_6]$ with Γ_{P_1} of order 2, generated by $g_1 = (v_1v_3)(v_2v_4)(v_5v_6)$ in cycle notation on the vertices of P_1 (v_7 and v_8 are mapped elsewhere). It interchanges the orientated faces $[v_1, v_3, v_4, v_6]$ and $[v_3, v_1, v_2, v_5]$ of P_1, and those faces have trivial stabilisers. The orientated face $[v_2, v_6, v_4, v_5]$ is mapped to itself by g_1 but its stabiliser is non-cyclic of order 4, with one of the two orientation reversing elements acting as $h_1 = (v_2v_5)(v_4v_6)$. The two triangles $[v_1, v_6, v_2]$ and $[v_3, v_5, v_4]$ are interchanged by g_1, and both have stabilisers of order 2, with the one for $[v_1, v_6, v_2]$ generated by (v_2v_6).

We have $P_2 = [v_1, v_8, v_3; v_2, v_7, v_5]$ with Γ_{P_2} of order 3, generated by $g_2 = (v_1v_3v_8)(v_2v_5v_7)$. The three orientated faces $[v_2, v_7, v_8, v_1]$, $[v_7, v_5, v_3, v_8]$ and $[v_5, v_2, v_1, v_3]$ are all in the same Γ-orbit and have trivial stabilisers. The two triangles $[v_1, v_8, v_3]$ and $[v_2, v_5, v_7]$ are necessarily non-conjugate (even ignoring orientation) as Γ_{P_2} has order 3, but both have (orientation non-preserving) stabiliser of order 6, which acts as the full permutation group on their vertices.

We now subdivide both P_1 and P_2 as stated just before (5.6). This gives an element

$$\begin{aligned} \pi_T' := \frac{1}{2}\pi_T = 3[v_3, v_1, v_2, v_6] - 3[v_3, v_5, v_2, v_6] + 3[v_3, v_5, v_4, v_6] \\ + 2[v_1, v_2, v_7, v_5] - 2[v_1, v_8, v_7, v_5] + 2[v_1, v_8, v_3, v_5]\,. \end{aligned}$$

Applying $f_{3,k}$ to this element results in

$$\beta' = 7[\tfrac{1}{3}\omega + \tfrac{2}{3}] - 3[-\tfrac{2}{3}\omega + \tfrac{5}{3}] + 3[\tfrac{1}{6}\omega + \tfrac{5}{6}] - 2[-\tfrac{1}{6}\omega + \tfrac{7}{6}]$$

in $\overline{\mathfrak{p}}(k)$.

Using (5.6) one can easily compute the boundary $\partial\pi_T'$. Because

$$\Sigma_2^* = \{(v_1, v_3, v_4, v_6), (v_2, v_6, v_4, v_5), (v_3, v_4, v_5), (v_1, v_8, v_3), (v_7, v_2, v_5)\}\,,$$

under the action of Γ we can move the resulting triangles to the eight triangles that result from the elements of Σ_2^*, with one of the 'squares' giving rise to four inequivalent triangles due to the two ways of triangulating a 'square', the other to only one inequivalent triangle.

In $(\oplus_t \mathbb{Z}[t])_\Gamma$, where t runs through the triangles, the triangular faces in $\partial(\pi'_T)$ coming from those of P_1 and P_2 cancel under the action of Γ (this uses that the coefficient of $[P_2]$ in π'_T is even and the triangular faces of P_2 have orientation reversing elements in their stabilisers). Of course, the 'internal' triangles created by the subdivision into tetrahedra always cancel. Using g_1, g_2, h_1, h_2 one moves the triangles coming from the 'square' faces to the five inequivalent triangles coming from $[v_1, v_3, v_4, v_6]$ and $[v_2, v_6, v_4, v_5]$. This yields the sum of the elements

$$\begin{aligned}\partial[v_4, v_3, v_1, v_6] = &\, 3([v_3, v_1, v_6] - [v_3, v_4, v_6] + [v_1, v_6, v_4] - [v_1, v_3, v_4]) \\ &+ 2([v_3, v_4, v_6] - [v_3, v_1, v_6] + 2([v_1, v_3, v_4] - [v_1, v_6, v_4]))\end{aligned}$$

and

$$3[v_5, v_4, v_6] - 3[v_5, v_2, v_6] = [v_5, v_4, v_6] - [v_5, v_2, v_6] - \partial[v_5, v_2, v_4, v_6]\,,$$

where we used h_1.

So for $\pi_T = 2\pi'_T = 6[P_1] + 4[P_2]$ we get $\partial(\pi_T - 2\partial[v_4, v_3, v_1, v_6] + 3\partial[v_5, v_2, v_4, v_6]) = 0$ modulo the action of Γ. After multiplying by 2 in order to deal with the ambiguity of the lifts of the cusps to $\mathcal{O}^2$, we then find the element

$$\beta_{\text{geo}} = 4\beta' - 4[3] + 6[4/5] \in \overline{B}(k),$$

with the last two terms arise because $\mathrm{cr}_3([v_4, v_3, v_1, v_6]) = 3$ and $\mathrm{cr}_3([v_5, v_2, v_4, v_6]) = 4/5$.

To see if one could do better, as discussed just before this example, we, instead, compute $\partial_{2,k}$ of β'. This can be done easily using the matching of triangles under the action of Γ as before, using the commutativity of (3.16) for $\partial_{2,k}^{\mathcal{O}^*}$, but as for this we have to lift the vertices to the column vectors in $\mathcal{O}^2$ in (5.5) (and not up to scaling by $\mathcal{O}^*$) we pick up some additional torsion along the way. Alternatively, one can choose a finite set of finite primes S for k such that, for every $[z]$ occurring in β', both z and $1-z$ are S-units, and compute in $\tilde{\wedge}^2 k^*$ as in Remark 3.9 and Corollary 3.8. The result is

$$-(-4)\tilde{\wedge}(5) + (-2)\tilde{\wedge}(3) + (-1)\tilde{\wedge}(2) + (-1)\tilde{\wedge}(-\omega) - 2\tilde{\wedge}(-5)$$

in $\tilde{\wedge}^2 k^*$. Note that $[v_5, v_4, v_6] - [v_5, v_2, v_6]$ under $f_{2,k}$ is mapped to $(-\frac{1}{2})\tilde{\wedge}(-\frac{\omega}{2}) - 2\tilde{\wedge}(-\omega) = (-1)\tilde{\wedge}(-\omega) - 2\tilde{\wedge}(-5)$. The first three terms are in the image of $\overline{\mathfrak{p}}(\mathbb{Q})$, and if we multiply the last by 2 then we obtain $-4\tilde{\wedge}(-5) = -(-4)\tilde{\wedge}(-5) = -(-4)\tilde{\wedge}5 + (-1)\tilde{\wedge}(-1)$, with the first again in the image of $\overline{\mathfrak{p}}(\mathbb{Q})$. If $(-1)\tilde{\wedge}(-1)$ would come from $\overline{\mathfrak{p}}(\mathbb{Q})$ then it would come from its torsion by Proposition 3.31(ii), which is generated by [2]. By Lemma 3.10, the kernel of $\tilde{\wedge}^2\mathbb{Q}^* \to \tilde{\wedge}^2 k^*$ has order 2, and is generated by $(-1)\tilde{\wedge}(-5)$, and one easily checks using Corollary 3.8 that both $(-1)\tilde{\wedge}(-1)$ and $(-1)\tilde{\wedge}(-1) + (-1)\tilde{\wedge}(-5)$ in $\tilde{\wedge}^2\mathbb{Q}^*$ are neither trivial nor equal to $\partial_{2,\mathbb{Q}}([2]) = (-1)\tilde{\wedge}2$. Therefore $(-1)\tilde{\wedge}(-1)$ in $\tilde{\wedge}^2 k^*$ is not in the image of $\overline{\mathfrak{p}}(\mathbb{Q})$. Multiplying by 2 again kills the term $(-1)\tilde{\wedge}(-1)$, hence $4\beta - 4[3] + 6[5]$ is in $\overline{B}(k)$ is the best possible for our choice of subdivision. (Note this element equals β_{geo} above as $[\frac{4}{5}] = -[\frac{1}{5}] = [5]$. Also note that these calculations also show that $2\beta' - 2\overline{\beta'}$ is in $\overline{B}(k)$, in line with Remark 5.4, and that this element must also equal β_{geo}.)

In fact, $K_2(\mathcal{O})$ is trivial by [2, §7], so by Corollary 4.10(i), $\psi_k(\beta_{\rm geo})$ is a generator of the infinite cyclic group $K_3(k)^{\rm ind} = K_3(k)^{\rm ind}_{\rm tf}$, $\beta_{\rm geo}$ is a generator of the infinite cyclic group $\overline{B}(k)$, and the map $\psi_k\colon \overline{B}(k) \to K_3(k)^{\rm ind}_{\rm tf} = K_3(k)^{\rm ind}$ is an isomorphism.

This shows that one cannot improve upon the above by using a different triangulation, and that the obstruction of 'incompatible lifts' under the action of Γ is non-trivial, so that both of the factors of 2 mentioned in Remark 5.3 are necessary in this case.

6. Finding a generator of $K_3(k)$, and computing $|K_2(\mathcal{O}_k)|$

In this final section, we again restrict to the case of an imaginary quadratic field k and set $\mathcal{O} := \mathcal{O}_k$. For simplicity of exposition, we fix an embedding $\sigma : k \to \mathbb{C}$ and use it to regard k as a subfield of $\mathbb{C}$.

We explain how to combine an implementation of an algorithm of Tate's, which produces a natural number that is known to be divisible by the order of $K_2(\mathcal{O})$, with either the result of Corollary 4.10(i) or just the known validity of the precise form of Lichtenbaum's conjecture for k and $m = 2$ (cf. Remark 2.11), to deduce our main computational results.

At this stage we have successfully applied the first of these approaches to about twenty fields and the second to hundreds of fields. In this way, for example, we have, for all imaginary quadratic fields of discriminant bigger than -1000, determined the order of $K_2(\mathcal{O})$, where not yet known, and a generator of the infinite cyclic group $K_3(k)^{\rm ind}_{\rm tf}$ that lies in the image of the injective homomorphism ψ_k constructed in Theorem 3.25 (thereby verifying that k validates Conjecture 3.34) and hence also the Beilinson regulator value $R_2(k)$.

The results are available at

`https://mathstats.uncg.edu/yasaki/data/`

In particular, for each of the listed imaginary quadratic number fields k, the element $\beta_{\rm alg}$ is such that its image $\psi_k(\beta_{\rm alg})$ generates $K_3(k)^{\rm ind}_{\rm tf}$, thus verifying Conjecture 3.34 for all those fields. The element $\beta_{\rm geo}$ is the element of Theorem 4.7, obtained in the way described in Remark 5.4.

6.1. Dividing $\beta_{\rm geo}$ by $|K_2(\mathcal{O}_k)|$.

6.1.1. The basic approach is as follows. An implementation by Belabas and Gangl [2] of (a refinement of) an algorithm of Tate gives an explicit natural number M which is known to be divisible by $|K_2(\mathcal{O})|$.

Since M is typically sharp as an upper bound on $|K_2(\mathcal{O})|$, for any element $\alpha_{\rm geo}$ of $\ker(delta_{2,k})$ that lifts $\beta_{\rm geo}$, we try to find an element α in this kernel for which the difference $M \cdot \alpha - \alpha_{\rm geo}$ lies in the subgroup generated by the relations (3.18) and (3.19). If one finds such an element α, then its class β in $\overline{B}(k)$ is such that $\beta_{\rm geo} = M \cdot \beta$. From the result of Corollary 4.10(i) it then follows that $|K_2(\mathcal{O})| = M$, that β generates $\overline{B}(k)$, that $\psi_k(\beta)$ generates $K_3(k)^{\rm ind}_{\rm tf}$ and hence, by Theorem 3.4(iii), that $R_2(k) = |{\rm reg}_2(\psi_k(\beta))|$.

6.1.2. To find a candidate element α as above we first use the methods described in §6.3 below to identify an element α for which one can verify *numerically* that $M \cdot \mathbb{D}_\sigma(\alpha) = \mathbb{D}_\sigma(\alpha_{\rm geo})$, with $\mathbb{D}_\sigma$ the homomorphism from Remark 3.21. We then aim to prove *algebraically* that $M \cdot \beta = \beta_{\rm geo}$ by writing the difference $M \cdot \alpha - \alpha_{\rm geo}$ as a sum of explicit relations of the form (3.18) and (3.19).

To complete this last step we use a strategy that can be used to investigate whether any element of the form $\sum_i n_i[x_i]$, where the n_i are integers and the x_i are in $k^\flat$, can be written as a sum of such relations, using suitable finite subsets U of $k^\flat$.

We first take U to consist of all elements x_i and their images under the 6-fold symmetries that are used in the relations (3.19), i.e., for u in U we also adjoin $1-u$, u^{-1}, $1-u^{-1}$, $(1-u^{-1})^{-1} = \frac{-u}{1-u}$ and $(1-u)^{-1}$ to U.

For $u \neq v$ in U, we then consider the element in $\mathbb{Z}[k^\flat]$ that is obtained by putting $x = u$ and $y = v$ in (3.18). We use only the result if all five terms are in U.

We then form a matrix A of width $|U|$, as follows.

- For the first row we write $\sum_i n_i[x_i]$ in terms of the $\mathbb{Z}$-basis $\{[u]$ with u in $U\}$ of the subgroup $\mathbb{Z}[U]$ of $\mathbb{Z}[k^\flat]$.
- For each of the, n say, 5-term relations that we have just generated, we add a row writing it in terms of the basis.
- For each u in U we add rows corresponding to the relations $[u] + [1-u]$, $[u] + [u^{-1}]$, $[u] - [1-u^{-1}]$, $[u] + [\frac{-u}{1-u}]$ and $[u] - [(1-u)^{-1}]$, resulting in, say, m rows in total.

Then the kernel of the right-multiplication by A on $\mathbb{Z}^{1+n+m}$ (regarded as row vectors) gives the relations among the various elements that we put into the rows of A. In particular, if we find an element in this kernel that has 1 as its first entry, then we have succeeded in writing $\sum_i n_i[x_i]$ explicitly in terms of the elements of the form (3.18) and (3.19).

Unfortunately, however, this rather straightforward method is rarely successful. Instead, we may have to allow increases in the set U, which may make the computation too large to be carried out. It was, however, done successfully, to some extent by trial and error, for a number of imaginary quadratic number fields.

Example 6.1. The most notable example among those is the field $k = \mathbb{Q}(\sqrt{-303})$, for which it is known from [2] that $|K_2(\mathcal{O})| = 22$. The results for this case are described in Appendix B.

Remark 6.2. We note that the method described above for verifying identities in $\overline{B}(k)$ only depends on the definition of $\overline{B}(k)$ in terms of the boundary map $\delta_{2,k}$ and the relations (3.18) and (3.19) on $\mathbb{Z}[k^\flat]$ that are used to define $\overline{\mathfrak{p}}(k)$. In particular, it does not rely on knowing the validity of Lichtenbaum's Conjecture and so, in principle, the same approach could be used to show that an element is trivial in $\overline{\mathfrak{p}}(F)$ for any number field F (although, in practice, the computations are likely to quickly become unfeasibly large).

6.2. **Finding a generator of $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$ directly.** This approach relies on the effective bounds on $|K_2(\mathcal{O})|$ that are discussed above, the known validity of Lichtenbaum's Conjecture as in Remark 2.11, and an implementation of the 'exceptional S-unit' algorithm (see §6.3 below) that produces elements in $\overline{B}(k)$. In particular, the reliance on Lichtenbaum's Conjecture means that the general applicability of this sort of approach is currently restricted to abelian fields.

To describe the basic idea, we assume to be given an element γ that equals N_γ times a generator of the (infinite cyclic) group $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$ for some non-negative integer N_γ.

Then one has $|\mathrm{reg}_2(\gamma)| = N_\gamma \cdot R_2(k)$ and so Remark 2.11 implies that

$$-\frac{\mathrm{reg}_2(\gamma)}{12 \cdot \zeta_k'(-1)} = \frac{N_\gamma}{|K_2(\mathcal{O})|}.$$

If one now also has an explicit natural number M that is known to be divisible by $|K_2(\mathcal{O})|$, then one knows that the quantity

$$-M\frac{\mathrm{reg}_2(\gamma)}{12\cdot\zeta_k'(-1)} = N_\gamma\frac{M}{|K_2(\mathcal{O})|} \tag{6.3}$$

is a product of a non-negative and a positive integer.

Hence, if one finds that the left hand side of this equality is numerically close to a natural number d_γ, then N_γ and $M/|K_2(\mathcal{O})|$ are both divisors of d_γ. In particular, if one can find an element γ for which $d_\gamma = 1$, then one conclude both that $N_\gamma = 1$ (so that γ is a generator of $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$) and that $|K_2(\mathcal{O})|= M$. We would therefore have identified a generator of $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$ and determined $|K_2(\mathcal{O})|$.

To find suitable test elements γ we use the method described in §6.3 below to generate elements α in the subgroup $\ker(\delta_{2,k})$ of $\mathbb{Z}[k^\flat]$. We then take γ to be the image under ψ_k of the image of α in $\overline{B}(k)$.

Note here that it is not *a priori* guaranteed that $\mathrm{im}(\psi_k)$ contains a generator of $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$. However, if this is the case (as it was in all of the examples we tested), then ψ_k is surjective and so one has verified that k validates Conjecture 3.34.

6.3. Constructing elements in $\ker(\delta_{2,k})$ via exceptional S-units.

6.3.1. In order to find enough elements in $\ker(\delta_{2,k})$ we fix a finite set S of finite places of k, and consider 'exceptional S-units', where an exceptional S-unit is an S-unit x such that $1-x$ is also an S-unit.

To compute with such elements it is convenient to fix a basis of the S-units of k, i.e., a set of S-units that give a $\mathbb{Z}$-basis of the S-units modulo torsion. (This is implemented in GP/PARI [44] as 'bnfsunits'.) We then encode exceptional S-units in terms of the exponents that arise when they are expressed in terms of the fixed basis, and a suitable root of unity. Using Corollary 3.8 and Remark 3.9 then enable us to compute effectively with the image in $\tilde{\wedge}^2 k^*$ of the elements $(1-x)\tilde{\wedge}x$ for the exceptional S-units x.

Example 6.4. In the case $k = \mathbb{Q}(\sqrt{-11})$ and $S = \{\wp_2, \wp_3, \overline{\wp}_3\}$ where $\wp_2 = (2)$ is the unique prime ideal of norm 4 and $\wp_3$ and $\overline{\wp}_3$ denote the two prime ideals of norm 3 in $\mathcal{O}$, PARI provides the S-unit basis $\mathcal{B} = \{b_1, b_2, b_3\}$ with $b_1 = 2$, $b_2 = \frac{-1+\sqrt{-11}}{2}$ and $b_3 = \frac{-1-\sqrt{-11}}{2}$. We find the exceptional S-unit $x = \frac{5}{36} - \frac{\sqrt{-11}}{36}$ of norm $\frac{1}{36}$, for which $1-x$ has norm $\frac{3}{4}$, and write

$$x = -b_1^{-1}b_2^{-2}, \qquad 1-x = -b_1^{-1}b_2^{-2}b_3^3 .$$

It follows that

$$(1-x)\tilde{\wedge}x = (-1)\tilde{\wedge}(-1) + (-1)\tilde{\wedge}b_1 + (-1)\tilde{\wedge}b_3 + 3(b_1\tilde{\wedge}b_3) + 6(b_2\tilde{\wedge}b_3),$$

which corresponds to the element $(\overline{1},\overline{1},\overline{0},\overline{1},0,3,6)$ under the isomorphism in Corollary 3.8.

This approach effectively reduces the problem of finding elements in $\ker(\delta_{2,k})$ to a concrete problem in linear algebra. Of course, one wants to choose a finite set S of finite places for which one can find sufficiently many exceptional S-units in k such that some linear combination of them in $\ker(\delta_{2,k})$ gives a *non-trivial* element in (and preferably a generator of) the quotient $\overline{B}(k)$.

Note that, whilst one can check for non-triviality of an element β in $\overline{B}(k)$ by simply verifying that its image under the map $\mathbb{D}$ is numerically non-trivial, in order to conclude that β is trivial, we need to know an explicit natural number M that is divisible by $|K_2(\mathcal{O})|$. Then the quantity on the left hand side of (6.3) is numerically close to zero if and only if $\gamma = \psi_k(\beta)$ is trivial. If that is the case, then the injectivity of ψ_k implies that the element β is itself trivial.

6.3.2. Since the map ψ_k has finite cokernel there always exists a finite set S that results in a non-trivial element $\overline{B}(k)$ by the above procedure.

In practical terms, the geometric approach in §4 shows that one need only take $S = S_{\text{geo}}$ to be the set comprising all of the places that divide any of the principal ideals $\mathcal{O} \cdot r_{i,j}$ and $\mathcal{O} \cdot (1 - r_{i,j})$ for the elements $r_{i,j}$ that occur in Theorem 4.7.

In general, this set S_{geo} is far too large to be practical for the exceptional S-unit approach. Fortunately, however, in all of the cases investigated in this paper we have found that a much smaller choice of set S is sufficient. In fact, we have found that it is often enough to take S to comprise all places that divide either 2 or 3 or any of the first ten (say) primes that split in k.

Example 6.5. In the case $k = \mathbb{Q}(\sqrt{-303})$ it suffices to take S to be the set of places that divide either of $2, 11$ and 13 (all of which split in k) or 3 (which ramifies in k). Imposing small bounds on the exponents with respect to a chosen basis, we already find 683 exceptional S-units in k. Setting $\omega := (1 + \sqrt{-303})/2$, GP/PARI's [44] 'bnfsunits' gives as a basis of the S-units the set of elements

$$\{-20 + 3\omega, 2, -4 - \omega, -36 - \omega, 4 - \omega, 28 - \omega, -12 + \omega\}$$

of norms 2^{10}, 2^2, $2^5 \cdot 3$, $2^7 \cdot 11$, $2^3 \cdot 11$, $2^5 \cdot 13$, and $2^4 \cdot 13$, respectively.

Then $\ker(\delta_{2,k})$ is a free $\mathbb{Z}$-module of rank several hundreds, but most of the elements of a $\mathbb{Z}$-basis for this kernel turn out to result in the trivial element of $\overline{\mathfrak{p}}(k)$, i.e., correspond to relations of the type (3.18) and (3.19). In this case, with a set of exceptional units that differed from the one used in Appendix B, but again using that $|K_2(\mathcal{O})|= 22$, we found using the approach described in §6.2 that the element

$$-46[\frac{-\omega-27}{64}] + 36[\frac{-\omega+15}{16}] - 14[\frac{-\omega+41}{16}] - 48[\frac{-\omega+8}{4}] - 62[\frac{-\omega+41}{4}] + 18[\frac{-\omega+41}{48}] + 34[\frac{-11}{2}]$$
$$+ 42[\frac{-143}{1}] - 16[\frac{-253\omega+2321}{6144}] + 28[\frac{-253\omega-495}{3328}] + 158[\frac{-3\omega+23}{32}] + 4[\frac{-39\omega-221}{512}] + 120[\frac{-5\omega+73}{64}]$$
$$+ 18[\frac{-5\omega+49}{416}] + 44[\frac{\omega+91}{64}] + 70[\frac{\omega+27}{64}] - 12[\frac{\omega+91}{128}] - 36[\frac{\omega+1}{16}] + 82[\frac{\omega-2}{2}] + 92[\frac{\omega+7}{32}]$$
$$- 36[\frac{\omega+40}{4}] - 22[\frac{\omega-8}{4}] - 116[\frac{11}{2}] + 58[\frac{11}{8}] - 34[\frac{13}{2}] - 84[\frac{13}{4}] + 34[\frac{3\omega+20}{8}] + 14[\frac{9\omega-17}{352}]$$

of $\mathbb{Z}[k^\flat]$ belongs to $\ker(\delta_{2,k})$, and that its image in $\overline{B}(k)$ is sent by ψ_k to a generator of $K_3(k)^{\text{ind}}_{\text{tf}}$.

Example 6.6. We now set $k = \mathbb{Q}(\sqrt{-4547})$, so that $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega = \frac{1+\sqrt{-4547}}{2}$. We recall that in [11] it was conjectured that $|K_2(\mathcal{O})|$ should be equal to 233. In fact, while the program developed in loc. cit. showed that $|K_2(\mathcal{O})|$ divides 233, the authors were unable to verify their conjecture since this would have required them to work in a cyclotomic extension of too high a degree.

By using the approach described in §6.2, we were now able to verify that $|K_2(\mathcal{O})|$ is indeed equal to 233 and, in addition, that the element

$$
\begin{aligned}
&132[\tfrac{-2\omega+5}{117}] - 2[\tfrac{-1}{12}] + 8[\tfrac{-2\omega-3}{1404}] - 6[\tfrac{-2\omega+5}{18}] - 14[\tfrac{-2\omega+1752}{19683}] - 2[\tfrac{-1}{2}] + 8[\tfrac{-2\omega-3}{27}] + 2[\tfrac{-1}{288}] \\
&+ 2[\tfrac{-1}{3}] - 24[\tfrac{-2\omega+1752}{3159}] - 2[\tfrac{-1}{36}] + 128[\tfrac{-2\omega+5}{39}] + 24[\tfrac{-2\omega-3}{4212}] + 74[\tfrac{-2\omega+5}{52}] + 12[\tfrac{-2\omega+5}{54}] \\
&- 12[\tfrac{-2\omega-840}{6591}] - 54[\tfrac{-\omega+421}{351}] + 40[\tfrac{-2\omega-3}{72}] - 16[\tfrac{-2\omega-3}{78}] + 12[\tfrac{-2\omega+5}{8}] - 10[\tfrac{-\omega+421}{468}] \\
&+ 2[\tfrac{-13}{16}] + 4[\tfrac{-13}{18}] - 6[\tfrac{-13}{24}] + 2[\tfrac{-13}{243}] - 2[\tfrac{-13}{3}] - 2[\tfrac{-13}{4}] - 8[\tfrac{-169}{324}] - 2[\tfrac{-4\omega-1680}{177957}] - 6[\tfrac{-208}{81}] \\
&- 2[-26] - 4[\tfrac{-26}{3}] + 2[\tfrac{-27}{4}] + 42[\tfrac{-3\omega+1263}{2197}] - 12[\tfrac{-31\omega+162}{13182}] + 22[\tfrac{-31\omega-131}{2197}] - 4[\tfrac{-5\omega+6}{64}] \\
&- 8[\tfrac{-16\omega+6736}{2197}] + 2[\tfrac{-16\omega-4523}{2197}] - 26[\tfrac{2\omega-5}{1404}] + 2[\tfrac{2\omega+3}{18}] + 2[\tfrac{1}{18}] - 4[\tfrac{\omega+875}{1053}] - 4[\tfrac{\omega+875}{1296}] \\
&- 24[\tfrac{2\omega-5}{27}] + 12[\tfrac{2\omega+1750}{3159}] - 2[\tfrac{1}{32}] + 14[\tfrac{2\omega-5}{351}] - 14[\tfrac{2\omega-5}{4212}] - 50[\tfrac{2\omega+3}{52}] - 78[\tfrac{2\omega+3}{54}] \\
&+ 4[\tfrac{2\omega-842}{6591}] + 38[\tfrac{\omega+420}{351}] - 30[\tfrac{2\omega-5}{72}] - 2[\tfrac{2\omega-5}{78}] - 14[\tfrac{2\omega+3}{8}] + 14[\tfrac{\omega+420}{4056}] - 6[\tfrac{\omega+420}{468}] \\
&- 6[117] - 2[\tfrac{13}{256}] - 2[\tfrac{13}{81}] - 4[\tfrac{169}{16}] - 14[\tfrac{169}{243}] + 4[\tfrac{169}{256}] - 10[\tfrac{3494\omega-13298}{2197}] - 16[\tfrac{4\omega-1684}{177957}] \\
&- 42[\tfrac{16\omega+4523}{8788}] + 2[\tfrac{243}{256}] + 2[\tfrac{26}{27}] - 2[\tfrac{26}{9}] + 4[\tfrac{3}{32}] - 16[\tfrac{3\omega+1260}{2197}] + 10[\tfrac{31\omega+131}{13182}] - 24[\tfrac{31\omega-162}{2197}] \\
&- 6[\tfrac{39}{2}] + 6[\tfrac{39}{8}] + 8[\tfrac{8\omega+3360}{351}] + 20[\tfrac{8\omega+3360}{9477}] - 30[\tfrac{10\omega+2}{1053}] - 38[\tfrac{5\omega+1}{54}] + 6[\tfrac{5\omega+1}{64}] - 12[\tfrac{5\omega-6}{78}] \\
&- 4[\tfrac{5\omega+1}{1053}] + 24[\tfrac{5\omega+1}{27}] + 8[\tfrac{10\omega-12}{729}] - 6[52] - 4[\tfrac{52}{81}] - 2[\tfrac{64}{81}] - 14[\tfrac{16\omega+4523}{4563}] + 4[\tfrac{841\omega-176104}{177957}]
\end{aligned}
$$

of $\mathbb{Z}[k^\flat]$ belongs to $\ker(\delta_{2,k})$, and that its image in $\overline{B}(k)$ is sent by ψ_k to a generator of $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$.

Appendix A. Orders of finite subgroups

In this appendix we again consider a fixed imaginary quadratic field, embedded into $\mathbb{C}$. The main aim of this subsection is to prove, in Corollary A.5, that the lowest common multiple of the orders of finite subgroups of $\mathrm{PGL}_2(\mathcal{O})$ is either 12 or 24, with the latter being the case only if $k = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$. The authors are not aware of a suitable reference for this in the literature, or in fact, of an explicit classification of types of finite subgroups of $\mathrm{PGL}_2(\mathcal{O})$ that do not lie in $\mathrm{PSL}_2(\mathcal{O})$. As this is not difficult, we include it for the sake of completeness.

Using the inclusions $\mathrm{PSL}_2(\mathcal{O}) \subset \mathrm{PGL}_2(\mathcal{O}) \subset \mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$, our arguments are based on the following classical result [20, Chap. 2, Th. 1.6] that goes back to Klein [29].

Proposition A.1. *A finite subgroup of* $\mathrm{PSL}_2(\mathbb{C})$ *is isomorphic to either a cyclic group of order* $m \geq 1$*, a dihedral group of order* $2m$ *with* $m \geq 2$*,* A_4*,* S_4 *or* A_5*. Further, all of these possibilities occur.*

It seems that the finite subgroups of $\mathrm{PSL}_2(\mathcal{O})$ have been studied more than those of $\mathrm{PGL}_2(\mathcal{O})$ even though it is harder to determine them.

We note that an element γ in $\mathrm{SL}_2(\mathcal{O})$ of finite order has a characteristic polynomial of the form $x^2 + ax + 1$ with a in $[-2, 2] \cap \mathcal{O}$ as the two roots must be roots of unity, and conjugate. Hence $a = \pm 2$, ± 1 or 0, from which it follows readily that the image $\overline{\gamma}$ in $\mathrm{PSL}_2(\mathcal{O})$ of γ must have order 1, 2 or 3.

In view of Proposition A.1, this limits the possibilities of a finite subgroup of $\mathrm{PSL}_2(\mathcal{O})$ to the cyclic groups of orders 1, 2, or 3, the dihedral groups of order 4 or 6, and A_4.

Cyclic groups of order 2 or 3 can be obtained already in the subgroup $\mathrm{PSL}_2(\mathbb{Z})$, generated by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ respectively.

The occurrence of the dihedral groups of order 4 and 6, and of A_4, in $\mathrm{PSL}_2(\mathcal{O})$ for $k = \mathbb{Q}(\sqrt{-d})$ with d a positive square-free integer, depends on the prime factorisation of d; see [33, Satz 6.8.].

We now consider finite subgroups of $\mathrm{PGL}_2(\mathcal{O})$. An element in $\mathrm{PGL}_2(\mathcal{O})$ of odd order is contained in $\mathrm{PSL}_2(\mathcal{O})$: if an odd power of its determinant is a square in $\mathcal{O}^*$, then so is its determinant. It follows that the only finite subgroups that can occur in $\mathrm{PGL}_2(\mathcal{O})$ but are not necessarily contained in $\mathrm{PSL}_2(\mathcal{O})$ are the cyclic groups of even order, the dihedral groups, and S_4. In order to obtain a complete answer, we first look at elements of finite order in $\mathrm{PGL}_2(k)$.

Lemma A.2. *Let γ be an element of finite order in $\mathrm{GL}_2(k)$, and $\overline{\gamma}$ its image in $\mathrm{PGL}_2(k)$. Write $\mathrm{ord}(\gamma)$ and $\mathrm{ord}(\overline{\gamma})$ for the respective orders of these elements.*

(i) For $k = \mathbb{Q}(\sqrt{-1})$, one has $\mathrm{ord}(\gamma) \in \{1, 2, 3, 4, 6, 8, 12\}$ and $\mathrm{ord}(\overline{\gamma}) \in \{1, 2, 3, 4\}$.
(ii) For $k = \mathbb{Q}(\sqrt{-2})$, one has $\mathrm{ord}(\gamma) \in \{1, 2, 3, 4, 6, 8\}$ and $\mathrm{ord}(\overline{\gamma}) \in \{1, 2, 3, 4\}$.
(iii) For $k = \mathbb{Q}(\sqrt{-3})$, one has $\mathrm{ord}(\gamma) \in \{1, 2, 3, 4, 6, 12\}$ and $\mathrm{ord}(\overline{\gamma}) \in \{1, 2, 3, 6\}$.
(iv) For all other k, one has $\mathrm{ord}(\gamma) \in \{1, 2, 3, 4, 6\}$ and $\mathrm{ord}(\overline{\gamma}) \in \{1, 2, 3\}$.

Proof. Assume γ has order n, and let $a(x)$ in $k[x]$ be its minimal polynomial, so that $a(x)$ divides $x^n - 1$. The statement is clear if $a(x)$ splits into linear factors, so we may assume $a(x)$ is irreducible in $k[x]$ and of degree 2. If $a(x)$ is in $\mathbb{Q}[x]$ then it is an irreducible factor in $\mathbb{Q}[x]$ of $x^n - 1$, necessarily the nth cyclotomic polynomial as the mth cyclotomic polynomial divides $x^m - 1$ if m divides n. So $\varphi(n) = 2$, and $n = 3$, 4 or 6. If $a(x)$ is not in $\mathbb{Q}[x]$ then $a(x)\overline{a(x)}$ is irreducible in $\mathbb{Q}[x]$, it must be the nth cyclotomic polynomial, and k must be a subfield of $\mathbb{Q}(\zeta_n)$. Then $\varphi(n) = 4$, so $n = 8$ and $k = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$, or $n = 12$ and $k = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$. (Note that $n = 5$ is excluded as $\mathbb{Q}(\zeta_5)$ contains no imaginary quadratic field.) The statement about the order of $\overline{\gamma}$ follows by taking into account the factorisation of $x^n - 1$ over $k[x]$. In general, the $2m$th cyclotomic polynomial divides $x^m + 1$. But for $k = \mathbb{Q}(\sqrt{-1})$ and $n = 12$, so $m = 6$, we also have $x^6 + 1 = (x^3 - i)(x^3 + i)$ in $k[x]$, and $a(x)$ divides one of those factors. □

Remark A.3. In [37], in the cell stabiliser calculation for $k = \mathbb{Q}(\sqrt{-3})$, the symbol $\mathcal{A}_4$ should be a dihedral group of order 12 in $\mathrm{PGL}_2(\mathcal{O})$, where the subgroup in $\mathrm{PSL}_2(\mathcal{O})$ is dihedral of order 6.

We can now determine the types of finite subgroups in $\mathrm{PGL}_2(\mathcal{O})$ that do not lie in $\mathrm{PSL}_2(\mathcal{O})$.

Proposition A.4. *Let G be a finite subgroup of $\mathrm{PGL}_2(\mathcal{O})$ that is not contained in $\mathrm{PSL}_2(\mathcal{O})$.*

(i) For k not equal to $\mathbb{Q}(\sqrt{-m})$ with $m = 1$, 2 or 3, G is isomorphic to a cyclic group of order 2, or a dihedral group of order 4 or 6. For $k = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$, the list has to be extended with a cyclic group of order 4, a dihedral group of order 8, and S_4. For $k = \mathbb{Q}(\sqrt{-3})$ the list has to be extended with a cyclic group of order 6, and a dihedral group of order 12.
(ii) All the groups listed in claim (i) occur.

Proof. We already observed that an element of finite odd order in $\mathrm{PGL}_2(\mathcal{O})$ is contained in $\mathrm{PSL}_2(\mathcal{O})$, so the groups listed in claim (i) are those that are not ruled out by combining Proposition A.1 with Lemma A.2.

It remains to show that all such groups occur. Various examples may of course exist in the literature, but for the sake of completeness we give some here. In fact, for S_4 we use the stabiliser of the single element in Σ_3^* in our calculations for $\mathbb{Q}(\sqrt{-1})$ respectively $\mathbb{Q}(\sqrt{-2})$ (see §5.2.5).

Cyclic examples. If u in $\mathcal{O}^*$ is not a square, then $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is not in $\mathrm{PSL}_2(\mathcal{O})$ and its order equals the order of u. This gives the required subgroups except for those of order 2 for $\mathbb{Q}(\sqrt{-1})$, and of order 4 for $\mathbb{Q}(\sqrt{-2})$. The former can be obtained by using $\left(\begin{smallmatrix} 0 & \sqrt{-1} \\ 1 & 0 \end{smallmatrix}\right)$, which is not in $\mathrm{PSL}_2(\mathcal{O})$ and has order 2, and the latter by using $\left(\begin{smallmatrix} 0 & \frac{1}{\sqrt{-2}} \\ 1 & \sqrt{-2} \end{smallmatrix}\right)$, which is not in $\mathrm{PSL}_2(\mathcal{O})$ and has order 4.

Dihedral examples. If u in $\mathcal{O}^*$ is not a square, and has order $m = 2$, 4 or 6, then $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ generate a dihedral group of order $2m$. With $u = -1$ this constructs a copy of D_4, except for $\mathbb{Q}(\sqrt{-1})$, but for this field we can use generators $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ \sqrt{-1} & 0 \end{smallmatrix}\right)$. Taking u a generator of $\mathcal{O}^*$ gives a copy of D_8 for $\mathbb{Q}(\sqrt{-1})$, and a copy of D_{12} for $\mathbb{Q}(\sqrt{-3})$.

For k not equal to $\mathbb{Q}(\sqrt{-1})$, a suitable copy of D_6 is generated by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, whereas for $\mathbb{Q}(\sqrt{-1})$ we can use $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \sqrt{-1} \\ 1+\sqrt{-1} & -1 \end{smallmatrix}\right)$. Finally, a copy of D_8 for $\mathbb{Q}(\sqrt{-2})$ is generated by $\left(\begin{smallmatrix} 0 & \frac{1}{\sqrt{-2}} \\ 1 & \sqrt{-2} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

S_4-examples. For $\mathbb{Q}(\sqrt{-1})$, the orders of $\left(\begin{smallmatrix} 1 & \sqrt{-1}+1 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \sqrt{-1} & -1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \sqrt{-1} \\ 0 & -\sqrt{-1} \end{smallmatrix}\right)$ are 2, 3, and 4, respectively, and they generate a subgroup isomorphic to S_4.

For $\mathbb{Q}(\sqrt{-2})$, the orders of $\left(\begin{smallmatrix} -2 & -\sqrt{-2}-1 \\ -\sqrt{-2}+1 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -\sqrt{-2}-1 & -\sqrt{-2}+1 \\ 1 & \sqrt{-2} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & \sqrt{-2}+1 \\ \sqrt{-2}-1 & -2 \end{smallmatrix}\right)$ are 3, 3, and 2, respectively, and they also generate a subgroup isomorphic to S_4. □

Corollary A.5. *The lowest common multiple of the orders of the finite subgroups of* $\mathrm{PGL}_2(\mathcal{O})$ *is* 24 *if k is either* $\mathbb{Q}(\sqrt{-1})$ *or* $\mathbb{Q}(\sqrt{-2})$ *and is* 12 *in all other cases.*

Proof. This is true for the groups listed in Proposition A.4, and the possible finite subgroups of $\mathrm{PSL}_2(\mathcal{O})$ (discussed before Lemma A.2) have order dividing 12. □

Appendix B. A generator of $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}}$ for $k = \mathbb{Q}(\sqrt{-303})$

For an imaginary quadratic field $k = \mathbb{Q}(\sqrt{-d})$, with ring of integers $\mathcal{O}$, Browkin [10] has identified conditions under which the order $|K_2(\mathcal{O})|$ is divisible by either 2 or 3 (for example, he shows that $|K_2(\mathcal{O})|$ is divisible by 3 if $d \equiv 3 \mod 9$).

Moreover, all of the coefficients in the linear combination β_{geo} that occurs in Theorem 4.7(i) are divisible by 2 if k is not equal to either $\mathbb{Q}(\sqrt{-1})$ of $\mathbb{Q}(\sqrt{-2})$. In addition, if k is also not equal to $\mathbb{Q}(\sqrt{-3})$, then whilst Remark 5.3 shows that at least one of the coefficients in β_{geo} is not divisible by 3 one finds, in practice, that most of these coefficients are divisible by 3.

For these reasons, it can be relatively easy to divide β_{geo} by either 2 or 3. However, no such arguments work when considering division by primes larger than 3 and this always requires considerably more work.

It follows that if one uses the approach of §6.1, then any attempt to obtain a solution β in $\overline{\mathfrak{p}}(k)$, to the equation $|K_2(\mathcal{O})|{\cdot}\beta = \beta_{\mathrm{geo}}$, or equivalently (taking advantage of Remark 4.9) to the equation $2|K_2(\mathcal{O})|{\cdot}\beta = 2\beta_{\mathrm{geo}}$, obtaining a generator of $\overline{B}(k)$ is likely to be much more difficult when $|K_2(\mathcal{O})|$ is divisible by a prime larger than 3. This observation motivates us to discuss in some detail the field $\mathbb{Q}(\sqrt{-303})$.

For the rest of this section we therefore set

$$k := \mathbb{Q}(\sqrt{-303}),$$

so that $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega = (1 + \sqrt{-303})/2$.

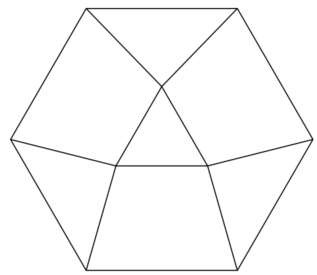

FIGURE B.1. Hexagonal cap

We recall that it was first conjectured in [11], and then verified in [2], that this field is the imaginary quadratic field of largest square-free discriminant for which $|K_2(\mathcal{O})|$ is divisible by a prime larger than 3. More precisely, this order was first conjectured and later determined to be equal to 22.

We apply the technique described in §4. The quotient $\mathrm{PGL}_2(\mathcal{O})\backslash\mathbb{H}$ has volume

$$\mathrm{vol}(\mathrm{PGL}_2(\mathcal{O})\backslash\mathbb{H}) = -\pi \cdot \zeta_k'(-1) \approx 140.17297686019148798153821412 15\ldots .$$

The tessellation of $\mathbb{H}$ consists of 132 distinct $\mathrm{PGL}_2(\mathcal{O})$-orbits of 3-dimensional polytopes:

- 87 tetrahedra
- 29 square pyramids
- 13 triangular prisms
- 1 octahedron
- 2 hexagonal caps—a polytope with a hexagonal base, 4 triangular faces, and 3 quadrilateral faces as shown in Figure B.1.

The stabiliser Γ_P in $\mathrm{PGL}_2(\mathcal{O})$ of each of these polytopes P is trivial except for eight polytopes P. It has order 2 for four triangular prisms, and order 3 for one triangular prism, the octahedron, and both hexagonal caps. By Theorem 4.7, the tessellation and stabiliser data give rise to an explicit element β_{geo}, which we compute by using the 'conjugation trick' of Remark 4.9 as $\frac{1}{2}(2\beta_{\mathrm{geo}})$ (see Remark 5.4 for why we are allowed to divide by 2). The latter can be written as a sum of 188 terms $a_i[z_i]$, where a_i is in $2\mathbb{Z}$ and z_i in $k^\flat$.

By Theorem 4.7 we have

$$\sum a_i D(z_i) = 24\pi \cdot \mathrm{vol}(\mathrm{PGL}_2(\mathcal{O})\backslash\mathbb{H}),$$

where D is the Bloch-Wigner dilogarithm.

Using the algebraic approach described in § 6.1, we can find an element in $\overline{B}(k)$ with image under the Bloch-Wigner function bounded away from 0, and it turns out that it suffices to restrict the search to exceptional S-units where S consists of the prime ideals above $\{2, 3, 11, 13, 19\}$. Here 3 ramifies in $\mathcal{O}$, and the other primes are the first four primes that split in $\mathcal{O}$. One of the combinations β_{alg} found with smallest positive dilogarithm value is a sum of 110 terms, $\beta_{\mathrm{alg}} = \sum b_j[z_j]$, where the b_j all happen to be ± 2.

By comparing $\sum b_j D(z_j)$ with $\sum_i a_i D(z_i)$ above, we expect that $\alpha = \beta_{\mathrm{geo}} - 22 \cdot \beta_{\mathrm{alg}}$ should represent the zero element in $\overline{B}(k)$. We can prove this by writing α explicitly as a sum of the elements specified in (3.18) and (3.19). A linear algebra calculation in Magma [9] shows that α can be written as an integral linear combination of 1648 specializations of the 5-term relation, plus a good number of 2-term relations. These relations are only valid up to torsion elements in $B(k)$ but are precisely valid in the torsionfree (by Corollary 3.30) group $\overline{B}(k)$. Hence we can conclude that $\beta_{\text{geo}} = 22 \cdot \beta_{\text{alg}}$, as required.

It follows from Corollary 4.10(i) that the resulting element $\psi_k(\beta_{\text{alg}})$ is a generator of $K_3(k)_{\text{tf}}$. Note that this also shows that the homomorphsm $\psi_k\colon \overline{B}(k) \to K_3(k)_{\text{tf}}^{\text{ind}}$ is bijective (as predicted by Conjecture 3.34).

The elements β_{alg} and β_{geo} are given below. The five-term combinations are given explicitly at `https://mathstats.uncg.edu/yasaki/data/` .

$$
\begin{aligned}
\beta_{\text{alg}} = &-2[\tfrac{-\omega+41}{52}] - 2[\tfrac{-\omega+2}{6}] - 2[\tfrac{-\omega-1}{6}] - 2[\tfrac{-\omega+92}{64}] - 2[\tfrac{-\omega+701}{676}] - 2[\tfrac{-\omega+12}{8}] - 2[\tfrac{-\omega+4}{8}] - 2[\tfrac{-\omega-3}{1}] - 2[\tfrac{-\omega+8}{12}] \\
&- 2[\tfrac{-\omega+12}{16}] - 2[\tfrac{-\omega+15}{16}] - 2[\tfrac{-\omega+2}{16}] - 2[\tfrac{-\omega-11}{16}] - 2[\tfrac{-\omega-25}{2}] - 2[\tfrac{-\omega+15}{22}] - 2[\tfrac{-\omega+26}{22}] - 2[\tfrac{-3\omega+46}{66}] \\
&- 2[\tfrac{-\omega-14}{22}] - 2[\tfrac{-\omega+26}{24}] - 2[\tfrac{-\omega+2}{26}] - 2[\tfrac{-\omega+8}{32}] - 2[\tfrac{-\omega+25}{32}] - 2[\tfrac{-\omega+28}{32}] - 2[\tfrac{-\omega-4}{32}] - 2[\tfrac{-\omega+389}{352}] \\
&- 2[\tfrac{-\omega-36}{352}] - 2[\tfrac{-\omega+8}{4}] - 2[\tfrac{-\omega-7}{4}] - 2[\tfrac{-\omega+41}{44}] - 2[\tfrac{-\omega+8}{48}] - 2[\tfrac{-15\omega+147}{104}] - 2[\tfrac{-4\omega+21}{13}] - 2[\tfrac{-4\omega+21}{33}] \\
&- 2[\tfrac{-21\omega+172}{352}] - 2[\tfrac{-21\omega+201}{352}] - 2[\tfrac{-23\omega+211}{256}] - 2[\tfrac{-23\omega+124}{312}] - 2[\tfrac{-2457\omega-611}{22528}] - 2[\tfrac{-253\omega+2321}{6144}] \\
&- 2[\tfrac{-253\omega-495}{3328}] - 2[\tfrac{-27\omega+535}{676}] - 2[\tfrac{-27\omega+168}{676}] - 2[\tfrac{-27\omega+535}{832}] - 2[\tfrac{-29\omega+1332}{1331}] - 2[\tfrac{-29\omega+149}{484}] \\
&- 2[\tfrac{-3\omega+84}{64}] - 2[\tfrac{-3\omega+45}{88}] - 2[\tfrac{-3\omega+46}{88}] - 2[\tfrac{-3\omega-9}{11}] - 2[\tfrac{-3\omega+36}{16}] - 2[\tfrac{-3\omega+45}{22}] - 2[\tfrac{-3\omega+46}{22}] \\
&- 2[\tfrac{-3\omega-20}{22}] - 2[\tfrac{-3\omega-21}{22}] - 2[\tfrac{-3\omega-17}{256}] - 2[\tfrac{-3\omega+15}{32}] - 2[\tfrac{-3\omega+24}{44}] - 2[\tfrac{-39\omega-221}{512}] - 2[\tfrac{-8\omega+31}{39}] \\
&- 2[\tfrac{-51\omega+3807}{3328}] - 2[\tfrac{-12\omega+63}{143}] - 2[\tfrac{-9\omega+17}{26}] + 2[\tfrac{\omega+11}{8}] + 2[\tfrac{\omega+3}{8}] + 2[\tfrac{\omega-4}{8}] + 2[\tfrac{\omega+25}{1}] + 2[\tfrac{\omega+14}{11}] \\
&+ 2[\tfrac{\omega+7}{12}] + 2[\tfrac{\omega+11}{16}] + 2[\tfrac{\omega+14}{16}] + 2[\tfrac{\omega+1}{16}] + 2[\tfrac{\omega+4}{16}] + 2[\tfrac{\omega-25}{16}] + 2[\tfrac{\omega+14}{22}] + 2[\tfrac{\omega+25}{24}] + 2[\tfrac{\omega-2}{24}] + 2[\tfrac{\omega+14}{26}] \\
&+ 2[\tfrac{\omega-2}{3}] + 2[\tfrac{\omega+4}{32}] + 2[\tfrac{\omega+7}{4}] + 2[\tfrac{\omega-4}{4}] + 2[\tfrac{\omega+11}{48}] + 2[\tfrac{\omega+7}{48}] + 2[\tfrac{15\omega+132}{104}] + 2[\tfrac{15\omega+204}{176}] + 2[\tfrac{21\omega-3}{169}] \\
&+ 2[\tfrac{21\omega+151}{352}] + 2[\tfrac{21\omega+180}{352}] + 2[\tfrac{23\omega+45}{256}] + 2[\tfrac{27\omega-51}{484}] + 2[\tfrac{29\omega+335}{512}] + 2[\tfrac{29\omega-1}{176}] + 2[\tfrac{3\omega-123}{121}] + 2[\tfrac{3\omega+33}{13}] \\
&+ 2[\tfrac{3\omega+33}{16}] + 2[\tfrac{3\omega+43}{22}] + 2[\tfrac{3\omega+3}{26}] + 2[\tfrac{3\omega+12}{32}] + 2[\tfrac{3\omega+17}{32}] + 2[\tfrac{3\omega+20}{32}] + 2[\tfrac{3\omega+20}{44}] + 2[\tfrac{33\omega+2651}{5408}] \\
&+ 2[\tfrac{8\omega+23}{39}] + 2[\tfrac{6399\omega+16348}{42592}] + 2[\tfrac{7\omega-67}{64}] + 2[\tfrac{7\omega-1}{66}] + 2[\tfrac{7\omega+181}{312}] + 2[\tfrac{9\omega+60}{52}] + 2[\tfrac{9\omega+8}{26}] + 2[\tfrac{9\omega+8}{44}].
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\text{geo}} = {} & 108[\tfrac{\omega+4}{6}] + 36[\tfrac{\omega+3}{5}] + 108[\tfrac{\omega+11}{13}] + 36[\tfrac{\omega+1}{5}] + 64[\tfrac{\omega+3}{4}] + 24[\tfrac{\omega+14}{26}] + 12[\tfrac{3\omega-3}{35}] + 36[\tfrac{3\omega}{38}] + 64[\tfrac{\omega}{4}] + 180[\tfrac{\omega+3}{8}] \\
& + 12[\tfrac{\omega-4}{22}] + 24[\tfrac{\omega+36}{32}] + 88[\tfrac{\omega+5}{10}] + 36[\tfrac{21\omega+192}{689}] + 24[\tfrac{\omega-4}{8}] + 136[\tfrac{3\omega+17}{32}] + 180[\tfrac{\omega+3}{11}] + 12[\tfrac{5\omega+44}{104}] + 30[\tfrac{9\omega+45}{106}] \\
& + 12[\tfrac{\omega-6}{2}] + 88[\tfrac{5\omega+23}{48}] + 20[\tfrac{5\omega-25}{3}] + 12[\tfrac{9\omega+38}{38}] + 12[\tfrac{\omega+19}{20}] + 48[\tfrac{\omega+4}{24}] + 24[\tfrac{5\omega-25}{128}] + 24[\tfrac{5\omega+148}{128}] + 60[\tfrac{\omega+24}{26}] \\
& + 24[\tfrac{42\omega+665}{1121}] + 24[\tfrac{6\omega+95}{77}] + 12[\tfrac{2\omega+17}{33}] + 12[\tfrac{\omega-1}{3}] + 48[\tfrac{\omega-1}{19}] + 24[\tfrac{7\omega-28}{55}] + 48[\tfrac{35\omega+644}{984}] + 24[\tfrac{4\omega+24}{59}] \\
& + 12[\tfrac{4\omega-28}{7}] + 24[\tfrac{7\omega+76}{55}] + 48[\tfrac{7\omega-28}{40}] + 60[\tfrac{\omega+3}{7}] + 12[\tfrac{\omega-5}{7}] + 24[\tfrac{5\omega-25}{56}] + 24[\tfrac{\omega+11}{9}] + 24[\tfrac{9\omega+99}{208}] + 60[\tfrac{4\omega+8}{41}] \\
& + 24[\tfrac{15\omega-4}{26}] + 36[\tfrac{\omega+27}{26}] + 36[\tfrac{\omega+27}{32}] + 12[\tfrac{2\omega+10}{53}] + 12[\tfrac{2\omega+41}{45}] + 12[\tfrac{\omega+75}{76}] + 6[\tfrac{\omega+5}{5}] + 6[\tfrac{5\omega}{81}] + 12[\tfrac{25\omega+1123}{1298}] \\
& + 12[\tfrac{11\omega+66}{118}] + 60[\tfrac{4\omega+29}{41}] + 24[\tfrac{15\omega+15}{26}] + 28[\tfrac{5\omega-28}{1272}] + 12[\tfrac{25\omega+1175}{1166}] + 12[\tfrac{\omega+5}{3}] + 72[\tfrac{3\omega+6}{41}] + 24[\tfrac{35\omega+440}{1007}] \\
& + 24[\tfrac{3\omega+20}{104}] + 24[\tfrac{\omega+58}{54}] + 24[\tfrac{9\omega+522}{583}] + 24[\tfrac{3\omega+20}{11}] + 12[\tfrac{\omega+37}{39}] + 12[\tfrac{6\omega+19}{247}] + 12[\tfrac{15\omega+980}{902}] + 12[\tfrac{\omega+7}{10}] \\
& + 24[\tfrac{3\omega-6}{13}] + 24[\tfrac{3\omega+57}{76}] + 24[\tfrac{\omega+36}{44}] + 12[\tfrac{\omega+76}{78}] + 12[\tfrac{27\omega+27}{130}] + 16[\tfrac{3\omega-21}{20}] + 28[\tfrac{3\omega+18}{59}] + 48[\tfrac{5\omega+67}{82}] \\
& + 28[\tfrac{15\omega+1370}{1298}] + 28[\tfrac{\omega+6}{10}] + 72[\tfrac{3\omega+32}{41}] + 12[\tfrac{8\omega+57}{209}] + 12[\tfrac{\omega+2}{3}] + 24[\tfrac{\omega+2}{7}] + 12[\tfrac{7\omega+64}{78}] + 12[\tfrac{16\omega+133}{53}] \\
& + 36[\tfrac{3\omega+15}{53}] + 24[\tfrac{\omega+4}{7}] + 48[\tfrac{2\omega+2}{39}] + 12[\tfrac{\omega-1}{2}] + 12[\tfrac{18\omega+684}{779}] + 12[\tfrac{9\omega-72}{308}] + 12[\tfrac{\omega+7}{7}] + 48[\tfrac{\omega+1}{4}] + 24[\tfrac{175\omega+1600}{4134}] \\
& + 30[\tfrac{9\omega+52}{106}] + 6[\tfrac{25\omega+867}{792}] + 24[\tfrac{7\omega+64}{50}]24[\tfrac{36\omega+1876}{2173}] + 24[\tfrac{\omega+40}{44}] + 24[\tfrac{3\omega+20}{14}] + 24[\tfrac{\omega+3}{44}] + 24[\tfrac{7\omega+137}{123}] \\
& + 6[\tfrac{27\omega-28}{26}] + 6[\tfrac{\omega+79}{82}] + 12[\tfrac{9\omega+55}{118}] + 12[\tfrac{16\omega+155}{779}] + 28[\tfrac{\omega+6}{6}] + 4[\tfrac{3\omega+17}{5}] + 16[\tfrac{30\omega+627}{1007}] + 24[\tfrac{4\omega-24}{9}] \\
& + 12[\tfrac{84\omega+1480}{2173}] + 24[\tfrac{9\omega-36}{28}] + 12[\tfrac{27\omega+988}{826}] + 12[\tfrac{7\omega+69}{76}] + 24[\tfrac{7\omega-28}{198}] + 12[\tfrac{21\omega+151}{130}] + 12[\tfrac{\omega+84}{88}] + 6[\tfrac{81\omega+331}{574}] \\
& + 12[\tfrac{9\omega+113}{104}] + 24[\tfrac{4\omega+31}{59}] + 16[\tfrac{11\omega+40}{106}] + 8[\tfrac{165\omega-765}{11236}] + 8[\tfrac{11\omega-51}{15}] + 36[\tfrac{21\omega+476}{689}] + 28[\tfrac{3\omega+274}{295}] + 28[\tfrac{5\omega+30}{59}] \\
& + 12[\tfrac{9\omega-54}{7}] + 8[\tfrac{5\omega-20}{9}] + 8[\tfrac{9\omega-36}{440}] + 4[\tfrac{\omega-4}{15}] + 4[\tfrac{5\omega+95}{152}] + 4[\tfrac{3\omega+35}{50}] + 4[\tfrac{\omega+19}{25}] + 16[\tfrac{10\omega+209}{159}] + 20[\tfrac{\omega-5}{160}] \\
& + 28[\tfrac{\omega+5}{265}] - 12[\tfrac{-3\omega}{35}] - 12[\tfrac{-\omega-3}{22}] - 12[\tfrac{-5\omega+49}{104}] - 12[\tfrac{-\omega-5}{2}] - 12[\tfrac{-9\omega+47}{38}] - 12[\tfrac{-\omega+20}{20}] - 12[\tfrac{-2\omega+19}{33}] \\
& - 12[\tfrac{-\omega}{3}] - 12[\tfrac{-4\omega-24}{7}] - 12[\tfrac{-\omega-4}{7}] - 12[\tfrac{-2\omega+12}{53}] - 12[\tfrac{-2\omega+43}{45}] - 6[\tfrac{-\omega+6}{5}] - 6[\tfrac{-5\omega+5}{81}] - 12[\tfrac{-25\omega+1148}{1298}] \\
& - 12[\tfrac{-11\omega+77}{118}] - 12[\tfrac{-25\omega+1200}{1166}] - 12[\tfrac{-\omega+6}{3}] - 12[\tfrac{-\omega+38}{39}] - 12[\tfrac{-6\omega+25}{247}] - 12[\tfrac{-15\omega+995}{902}] - 12[\tfrac{-\omega+8}{10}] \\
& - 12[\tfrac{-\omega+77}{78}] - 12[\tfrac{-27\omega+54}{130}] - 16[\tfrac{-3\omega-18}{20}] - 12[\tfrac{-8\omega+65}{209}] - 12[\tfrac{-\omega+3}{3}] - 12[\tfrac{-7\omega+71}{78}] - 12[\tfrac{-16\omega+149}{53}] \\
& - 12[\tfrac{-\omega}{2}] - 12[\tfrac{-18\omega+702}{779}] - 12[\tfrac{-9\omega-63}{308}] - 12[\tfrac{-\omega+8}{7}] - 6[\tfrac{-25\omega+892}{792}] - 6[\tfrac{-27\omega-1}{26}] - 6[\tfrac{-\omega+80}{82}] - 12[\tfrac{-9\omega+64}{118}] \\
& - 12[\tfrac{-16\omega+171}{779}] - 4[\tfrac{-3\omega+20}{5}] - 12[\tfrac{-84\omega+1564}{2173}] - 12[\tfrac{-27\omega+1015}{826}] - 12[\tfrac{-7\omega+76}{76}] - 12[\tfrac{-21\omega+172}{130}] - 12[\tfrac{-\omega+85}{88}] \\
& - 6[\tfrac{-81\omega+412}{574}] - 12[\tfrac{-9\omega+122}{104}] - 12[\tfrac{-9\omega-45}{7}] - 4[\tfrac{-\omega-3}{15}] - 4[\tfrac{-5\omega+100}{152}] - 4[\tfrac{-3\omega+38}{50}] - 4[\tfrac{-\omega+20}{25}];
\end{aligned}
$$

References

[1] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai, *Smooth compactifications of locally symmetric varieties*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, With the collaboration of Peter Scholze.

[2] K. Belabas and H. Gangl, *Generators and relations for $K_2\mathcal{O}_F$*, *K*-Theory **31** (2004), no. 3, 195–231.

[3] D. Benois and T. Nguyen Quang Do, *Les nombres de Tamagawa locaux et la conjecture de Bloch et Kato pour les motifs $\mathbb{Q}(m)$ sur un corps abélien*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 5, 641–672.

[4] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift I (Boston), Prog. in Math., vol. 86, Birkhäuser, 1990, pp. 333–400.

[5] S. J. Bloch, *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*, CRM Monograph Series, vol. 11, American Mathematical Society, Providence, RI, 2000, Manuscript from 1978 ("Irvine notes"). Published as volume 11 of CRM Monographs Series by American Mathematical Society.

[6] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. ENS **4** (1974), 235–272.

[7] ______, *Cohomologie de* SL_n *et valeurs de fonctions zêta aux points entiers*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), no. 4, 613–636, Errata in vol. 7, p. 373 (1980).
[8] ______, *Commensurability classes and volumes of hyperbolic* 3*-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), no. 1, 1–33.
[9] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265.
[10] J. Browkin, *The functor* K_2 *for the ring of integers of a number field*, Universal algebra and applications (Warsaw, 1978), Banach Center Publ., vol. 9, PWN, Warsaw, 1982, pp. 187–195.
[11] J. Browkin and H. Gangl, *Tame and wild kernels of quadratic imaginary number fields*, Math. Comp. **68** (1999), no. 225, 291–305.
[12] J. I. Burgos Gil, *The regulators of Beilinson and Borel*, CRM Monograph Series, vol. 15, Amer. Math. Soc., Providence, RI, 2002.
[13] D. Burns and M. Flach, *On Galois structure invariants associated to Tate motives*, Amer. J. Math. **120** (1998), no. 6, 1343–1397.
[14] ______, *Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. **6** (2001), 501–570 (electronic).
[15] D. Burns and C. Greither, *On the equivariant Tamagawa number conjecture for Tate motives*, Invent. Math. **153** (2003), no. 2, 303–359.
[16] D. Burns, R. de Jeu, and H. Gangl, *On special elements in higher algebraic* K*-theory and the Lichtenbaum-Gross conjecture*, Adv. Math. **230** (2012), no. 3, 1502–1529.
[17] R. de Jeu, *Zagier's conjecture and wedge complexes in algebraic* K*-theory*, Compositio Mathematica **96** (1995), 197–247.
[18] M. Dutour Sikirić, H. Gangl, P. E. Gunnells, J. Hanke, A. Schürmann, and D. Yasaki, *On the cohomology of linear groups over imaginary quadratic fields*, J. Pure Appl. Algebra **220** (2016), no. 7, 2564–2589.
[19] W. Dwyer and E. Friedlander, *Algebraic and étale* K*-theory*, Trans. Amer. Math. Soc. **292** (1985), no. 1, 247–280.
[20] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups acting on hyperbolic space: Harmonic analysis and number theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
[21] M. Flach, *On the cyclotomic main conjecture at the prime* 2, J. Reine Angew. Math. **661** (2011), 1–36.
[22] J.-M. Fontaine and B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions* L, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 599–706.
[23] J.-M. Fontaine, *Valeurs spéciales des fonctions* L *des motifs*, Astérisque (1992), no. 206, Exp. No. 751, 4, 205–249, Séminaire Bourbaki, Vol. 1991/92.
[24] A. B. Goncharov, *Polylogarithms and motivic Galois groups*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 43–96.
[25] E. Hecke, *Vorlesungen über die Theorie der algebraischen Zahlen*, Chelsea Publishing Co., Bronx, N.Y., 1970, Second edition of the 1923 original, with an index.
[26] A. Huber and G. Kings, *Bloch-Kato conjecture and Main Conjecture of Iwasawa theory for Dirichlet characters*, Duke Math. J. **119** (2003), no. 3, 393–464.
[27] A. Huber and J. Wildeshaus, *Classical motivic polylogarithm according to Beilinson and Deligne*, Doc. Math. **3** (1998), 27–133 (electronic), Erratum same volume, pages 297–299.
[28] B. Kahn, *The Quillen-Lichtenbaum conjecture at the prime 2*, preprint, 1997; available from `http://www.math.uiuc.edu/K-theory/208`.
[29] F. Klein, *Ueber binäre Formen mit linearen Transformationen in sich selbst*, Math. Ann. **9** (1875), no. 2, 183–208.
[30] F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55.
[31] M. Koecher, *Beiträge zu einer Reduktionstheorie in Positivitätsbereichen. I*, Math. Ann. **141** (1960), 384–432.
[32] M. Kolster, T. Nguyen Quang Do, and V. Fleckinger, *Twisted* S*-units,* p*-adic class number formulas, and the Lichtenbaum conjectures*, Duke Math. J. **84** (1996), no. 3, 679–717.

[33] N. Krämer, *Imaginärquadratische Einbettung von Ordnungen rationaler Quaternionenalgebren, und die nichtzyklischen endlichen Untergruppen der Bianchi-Gruppen*, preprint, 2015; available from `https://hal.archives-ouvertes.fr/hal-00720823/en/`.

[34] M. Le Floc'h, A. Movahhedi, and T. Nguyen Quang Do, *On capitulation cokernels in Iwasawa theory*, Amer. J. Math. **127** (2005), no. 4, 851–877.

[35] M. Levine, *The indecomposable* K_3 *of fields*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 2, 255–344.

[36] S. Lichtenbaum, *Values of zeta-functions, étale cohomology, and algebraic* K*-theory*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 489–501. Lecture Notes in Math., Vol. 342.

[37] E. R. Mendoza, *Cohomology of* PGL_2 *over imaginary quadratic integers*, Bonner Mathematische Schriften [Bonn Mathematical Publications], 128, Universität Bonn, Mathematisches Institut, Bonn, 1979, Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1979.

[38] D. Quillen, *Finite generation of the groups* K_i *of rings of algebraic integers*, Algebraic K-theory 1, Lecture Notes in Mathematics, vol. 341, Springer Verlag, Berlin, 1973, pp. 179–198.

[39] J. Rognes and C. Weibel, *Two-primary algebraic* K*-theory of rings of integers in number fields*, J. Amer. Math. Soc. **13** (2000), no. 1, 1–54, Appendix A by Manfred Kolster.

[40] P. Schneider, *Über gewisse Galoiscohomologiegruppen*, Math. Z. **168** (1979), no. 2, 181–205.

[41] C. Soulé, K*-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Inventiones Mathematicae **55** (1979), 251–295.

[42] V. Srinivas, *Algebraic* K*-theory*, second ed., Progress in Mathematics, vol. 90, Birkhäuser Boston Inc., Boston, MA, 1996.

[43] A. A. Suslin, K_3 *of a field, and the Bloch group*, Trudy Mat. Inst. Steklov. **183** (1990), 180–199, 229, Translated in Proc. Steklov Inst. Math. **1**991, no. 4, 217–239, Galois theory, rings, algebraic groups and their applications (Russian).

[44] The PARI Group, Univ. Bordeaux, *PARI/GP version* `2.11.0`, 2018, available from `http://pari.math.u-bordeaux.fr/`.

[45] G. Voronoi, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Premier mémoire. Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math. **133** (1908), 97–102.

[46] L. Washington, *Introduction to cyclotomic fields*, second ed., Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1997.

[47] C. Weibel, *Algebraic* K*-theory of rings of integers in local and global fields*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 139–190.

[48] ———, *The norm residue isomorphism theorem*, Topology **2** (2009), 346–372.

[49] C. Weibel, *Étale Chern classes at the prime* 2, Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 407, Kluwer Acad. Publ., Dordrecht, 1993, pp. 249–286.

[50] C. A. Weibel, *The* K*-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An introduction to algebraic K-theory.

[51] D. Yasaki, *Hyperbolic tessellations associated to Bianchi groups*, Algorithmic number theory, Lecture Notes in Comput. Sci., vol. 6197, Springer, Berlin, 2010, pp. 385–396.

[52] D. Zagier, *Polylogarithms, Dedekind zeta functions and the algebraic* K*-theory of fields*, Arithmetic algebraic geometry (Texel, 1989), Progr. Math., vol. 89, Birkhäuser Boston, Boston, MA, 1991, pp. 391–430.

King's College London, Dept. of Mathematics, London WC2R 2LS, United Kingdom
E-mail address: david.burns@kcl.acuk
URL: https://nms.kcl.ac.uk/david.burns/

Faculteit der Bètawetenschappen, Afdeling Wiskunde, Vrije Universiteit Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
E-mail address: r.m.h.de.jeu@vu.nl
URL: http://www.few.vu.nl/~jeu/

Department of Mathematical Sciences, South Road, Durham DH1 3LE, United Kingdom
E-mail address: herbert.gangl@durham.ac.uk
URL: http://maths.dur.ac.uk/~dma0hg/

Laboratoire de mathématiques GAATI, Université de la Polynésie française, BP 6570 — 98702 Faaa, French Polynesia
E-mail address: rahm@gaati.org
URL: https://gaati.org/rahm/

Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro, NC 27412, USA
E-mail address: d_yasaki@uncg.edu
URL: https://mathstats.uncg.edu/yasaki/