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Tahitian Pearls' Lustre Assessment

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Abstract

This paper represents a preliminary work toward a machine learning process which could be used to automatically assess Tahitian pearls' lustre. In particular, it investigates the different aspects of lustre which could be used to design feature vectors for machine learning algorithms.

Keywords: Pearl, Lustre, Features, Perception

1 Introduction

Tahitian pearls represent the first exportations of French Polynesia in terms of income[1]; this income is currently growing[2]. Millions of pearls are assessed each year by experts. On such a highly competitive market, an automated assistance could bring a crucial advantage to French Polynesia: under the RAPA[3] project, we undertook research on automatic measures of the thickness of nacre [3] and on characterizing pearls' colour [4], [5]. This paper presents our preliminary work on lustre automated assessment.

In section 2, we review the physical aspects of pearls, pointing out peculiarities of Tahitian pearls from an optical point of view, like the tendency to a stronger iridescence or darker colours than other pearls. In section 3, we address the notion of lustre and the distinction between its physical and perceptual dimensions, which do not necessarily correlate. The former consists of the optical phenomenon of specular reflection on nacre, while the latter consists of the perception of this phenomenon. In section 4, we review perceptual aspects of lustre. These aspects can serve to design feature vectors which could be used in a machine learning algorithm to reproduce the way about how experts assess lustre.

2 Physical and optical properties of nacre

A pearl is made of a nuclei coated with mother of pearl, or nacre, so that the physical aspects of its lustre borrow to the optical properties of its nacre. A complete view of nacre's structure can be found in [6]. Nacre is composed of aragonite plates (crystals about 0.5 μm thick) bound together by conchiolin (organic matter secreted by molluscs). These plates are structured in parallel layers; a quick computation shows that nacre contains about 2000 layers per millimetre thick. Aragonite is transparent and birefringent, whereas conchiolin contains pigments. Their structuration in parallel layers determines some optical properties of pearls [7]. As such, iridescence[4], sometimes called 'orient' in the specific case of pearls, is created by multiple reflections and refractions of light on the parallel interfaces formed by the interleaved layers of aragonite and conchiolin: the more the layers, the stronger

[1] 8,8 Billion F.CFP in 2014, representing 69% of the exportations of French Polynesia [1].
[2] More than 12% of growth between 2013 and 2014 [1].
[3] RAPA [2] stands for "Reconnaissance Automatique de la qualité des Perles de TAhiti".
[4] Iridescence (or 'goniochromism', or even sometimes referred to as 'perlescence') is the "interference of light either at the surface or in the interior of a material that produces a series of colours as the angle of incidence changes" [8]. This can be seen on thin layers of oil for example.

the ‘orient’. As well, aragonite plates are interleaved with conchiolin, yet conchiolin contains pigments, thus, the more the layers, the more the pigments, and darker are the pearls. Aragonite is a birefringent material; nevertheless, a camera fitted with a polarized filter does not allow detecting polarization due to birefringence on a pearl. The only polarization it detects stands on pearls’ boundaries when light is reflecting at grazing angle; however, this observation is explained through Fresnel’s equations and is not specific to pearls: the very same observation is done on artificial pearls. It is worth noting that nacre thickness is not necessarily uniform over the nucleus, thus lustre is not necessarily homogeneous over the surface of the pearl.
Tahitian pearls come from a mollusc called *pinctada margaritifera*. The conchiolin secreted by this mollusc contains black pigmentation [9]. Moreover, its nacre has the highest texture index [10] among other molluscs, resulting in more compact layers, that is, more layers per unit of nacre thickness. As a consequence, Tahitian pearls are darker and, for an equal thickness of nacre, exhibit a stronger orient than other pearls.

3 Notion of lustre

Definition: Lustre commonly refers either to an objective notion, as reflected high-lights on a surface[5], or to a subjective notion, as the perception we have of these reflected high-lights[6]. For the purpose of the present paper, let the former definition be qualified as ‘physical’ and the latter one as ‘perceptual’. Although synonym of gloss[7], ‘lustre’ is preferred to ‘gloss’ when one wants to stress the material out of which the reflecting surface is made. This usage is well illustrated in cases of minerals or fabrics. In the specific case of pearls, lustre is related to the layered structure of nacre[8]. In this regard, some pearls’ experts state that lustre is the ability to deeply reflect light, through these layers. The Gemological Institute of America (GIA) gives an incident definition of pearls’ lustre [16] by providing a list of adjectives that can qualify pearls’ reflections in order to assess it[9]. More specifically to Tahitian pearls, the *Assemblée de la Polynésie française*[10] (APF), in a normative text [17], states that the term ‘lustre’ can be replaced by ‘gloss’, and defines it as the “more or less perfect” reflection of light on the surface of the pearl. An excellent lustre is said to correspond to the total reflection of the light on the pearl’s surface and the ability to reflect images like a mirror, whereas no lustre would correspond to a matt finish of the surface. According to both the GIA and the APF, the quality of lustre depends on the physical parameters of the nacre: the GIA relates the poor quality of lustre to an insufficient thickness of the nacre, while the APF states that the quality of lustre depends on both the thickness and the structure of the nacre.
The notion of lustre may seem to be complex if not ambiguous (physical/perceptual duality), and the presence of a normative definition of Tahitian pearls’ lustre adds to the difficulty of deciding what definition to use. Nevertheless, *the goal when investigating Tahitian pearls’ lustre assessment is not to decide what the word ’lustre’ should mean but to capture what is actually done when a Tahitian pearl is being assessed regarding a characteristic called ‘lustre’*. From this point of view, the definition allowing capturing the most information about this process should be the broadest one, i.e., lustre is the *appearance* of *reflected high-lights on the surface of a pearl*. From this point of view, lustre is a particular realization of gloss, as defined by the *Compagnie Internationale de l’Eclairage* (CIE) [18].
Duality of lustre: Scientific insights on lustre can be found in the research literature about gloss. Even though this literature is not specific to lustre, and *a fortiori* to Tahitian pearls’ lustre, it applies to it since lustre is an instance of gloss. Gloss is commonly reduced to specular reflectance, and measuring instruments like glossmeters simply

[5] E.g., “the brightness that a shiny surface has” (Cambridge dictionary) [11], “the manner in which the surface … reflects light” (Oxford dictionary) [12].
[6] E.g., “the appearance of a … surface in terms of its light-reflective qualities” (Encyclopaedia Britannica) [13].
[7] E.g. [14], [15]
[8] As such, according to the Encyclopaedia Britannica, lustre “results from the repeated reflections from minute cleavage cracks” [13].
[9] ‘bright’, ‘sharp’ or ‘distinct’ for high lustre quality; ‘weak’, ‘hazy’, ‘blurred’, ‘dim’ or ‘diffused’ for low lustre quality
[10] Assembly of French Polynesia

measure specular reflectance at various angles[11]. Nonetheless, literature shows that physical and perceptual aspects of gloss evolve in distinct spaces that do not necessarily correlate. Indeed, despite common sense, gloss is not equivalent to specular reflectance. In other terms, specular reflectance is neither a sufficient nor even a necessary condition of gloss. First, the presence of specular reflectance is not sufficient to determine the presence of gloss. Indeed, materials exist for which specular reflectance is not perceived as gloss: snow for example, when seen at grazing angle, is not glossy, yet has a high specular reflectance[12]. Second, the presence of specular reflectance is not necessary to determine the presence of gloss. Indeed, it is possible to generate gloss using only diffuse light modulation [22]. To sum up, it is possible that specular reflectance be not perceived as gloss as well as that gloss be perceived despite the absence of specular reflectance. This demonstrates that gloss, hence lustre, cannot *a priori* be reduced to specular reflectance. Lustre has then to be studied as a perceptual phenomenon, and it is worth identifying the visual aspects coming into play in lustre assessment.

4 Aspects of lustre

Pearls exist as physical objects and, as such, exhibit optical properties. Yet they exist for an expert through its perceptual system and we have seen in the previous section that physical and perceptual properties do not necessarily coincide. The perceptual attributes of lustre have thus to be reviewed. These attributes, once extracted into feature vectors, can be used to automatically discover the underlying mechanisms of lustre perception. Hunter[23] provides a list of types of gloss: *specular gloss*, *sheen*, *contrast gloss*, *distinctness-of-reflected-image gloss*, *absence-of-bloom gloss* and *absence-of-surface-texture gloss*. We illustrate how these aspects are likely to allow grasping lustre as a perceptual phenomenon and emphasize the independence between them. In addition, we identify two aspects more specific to pearls; we call them *iridescence* and *deep reflectance*.

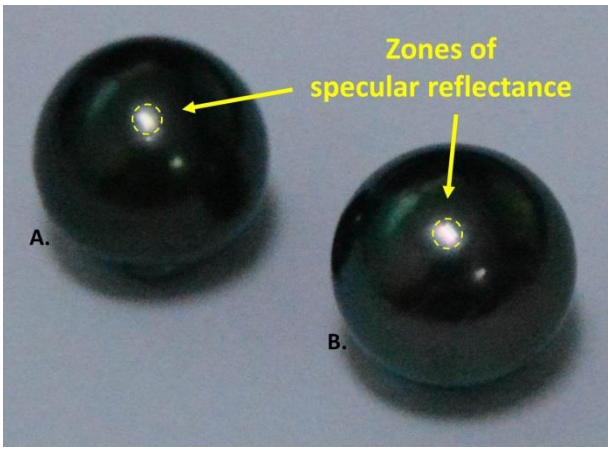


Figure 1: Two pearls of different *specular gloss* (maximal intensity values of the image in an HSV colour space: 93.33% for pearl A; 100% for pearl B).

Specular gloss: *Specular gloss* is the perceived brightness of a surface due to specular reflectance at non grazing angles with respect to incident light. Figure 1 shows two pearls of different *specular gloss*.

Numerical values corresponding to *specular gloss* could be the ratio of the incident intensity to the specular reflected intensity at non grazing angles or, if the incident intensity is constant, only the specular reflected intensity.

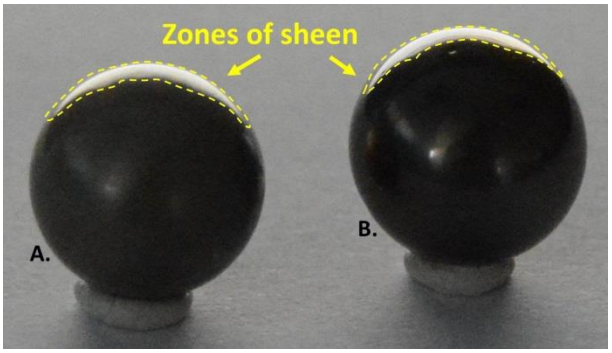


Figure 2: Two pearls of different *sheen* (maximal intensity values of the image in an HSV colour space: 96.47% for pearl A; 99.22% for pearl B).

Sheen: *Sheen* is the perceived brightness of a surface due to specular reflectance at grazing angles with respect to incident light. Figure 2 shows two pearls of different *sheen*. *Specular gloss* and *sheen*, though both expressing the strength of specular reflectance, are not necessarily dependent. Cases where there are both *specular gloss* and *sheen*, as well as cases where there is neither of them,

[11] For an example of glossmeter, see [19]

[12] [20] as cited by [21].

are commonly found. The case where there is *sheen* but no *specular gloss* is justified by Fresnel equations[13] and can be seen in flat wall paints. Eventually, the more counter intuitive case (because apparently contradicting Fresnel equations), where there *is specular gloss* but no *sheen*, has been found in "a number of yarn and paper samples which possessed a fuzziness that caused them to appear matt if viewed at near grazing angles" [23].

Numerical values corresponding to *sheen* can be obtained the same way as for *specular gloss*, but at grazing angles.

Contrast gloss: *Contrast gloss* is the perceived brightness of a surface due to the contrast between specular reflectance and diffuse reflectance. Figure 3 emphasizes the difference between *contrast gloss* and *specular gloss*: pearl B has a higher *contrast gloss* than pearl A[14], yet pearl A has a higher *specular gloss* than pearl B[15]. *Contrast gloss* and *specular gloss* both refer to the strength of reflected light at specular angle. However, they have different definitions: the former is defined with respect to incident light while the latter is defined with respect to diffuse reflectance. These two definitions express two different conceptions of gloss as a perception: *specular gloss* implies the capability of assessing specular reflectance given a reference that is not necessarily in the proximity of the specular angle, or even not directly accessible in the scene, while *contrast gloss* consists of assessing specular reflectance given a reference that is in the immediate proximity of the specular angle. In other words, unless considering *specular gloss* as mere specular reflectance[16], in which case its assessment would be based on references independent of the scene, *specular gloss* assessment should be based on actual or even estimated references from a global view of the scene, while *contrast gloss* assessment could be based only on actual references from a local view of the scene[17].

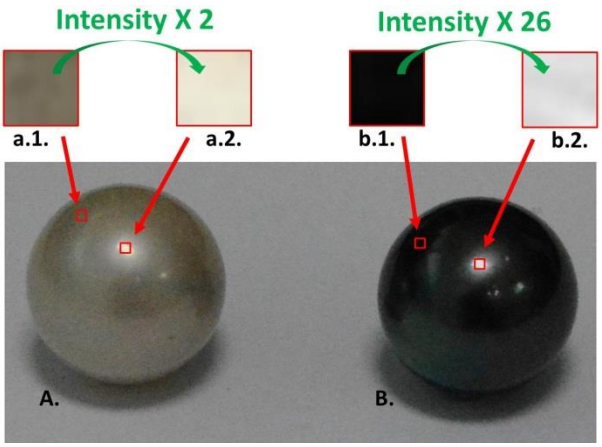


Figure 3: Two pearls of different *contrast gloss.* Pearl A has a lower contrast gloss than pearl B.

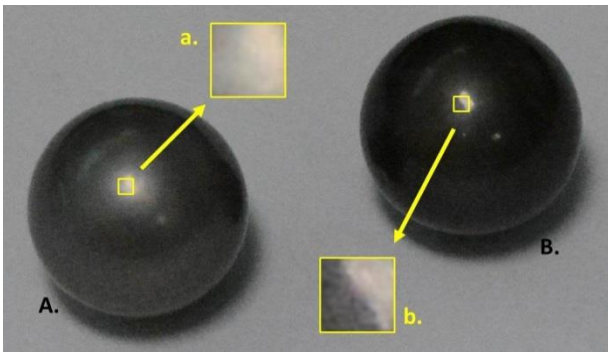


Figure 4: Two pearls of different *distinctness-of-reflected-image gloss.* Pearl A has a lower *distinctness-of-reflected-image gloss* than pearl B.

Numerical values corresponding to contrast gloss could be the ratio of specular reflected intensity to diffuse reflected intensity, or the ratio of the difference between specular and diffuse reflected intensities to the incident intensity.

Distinctness-of-reflected-image gloss: *Distinctness-of-reflected-image gloss* (DOI) is the perceived brightness of a surface due to the sharpness of the specular reflected light. The sharpest the specular reflected light, the more the surface behaves like a mirror. This aspect of gloss is sometimes called *mirror-like effect*. Figure 4 shows two pearls with different *distinctness-of-reflected-image gloss* qualities. The contour of the specular reflectance zone is

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[13] *Sheen* is related to the reflected light at grazing angle (large angle of incidence) while *specular gloss* is related to reflected light at non grazing angle (low angle of incidence); yet, according to Fresnel equations, the fraction of incident light that reflects from the surface is higher when the angle of incidence is larger [24].

[14] In an HSV colour space, the mean intensity value of a.2. is twice the one of a.1., while the one of b.2. is 26 times the one b.1.

[15] Maximal intensity values of the image in an HSV colour space: 93.73% for pearl A; 93.33% for pearl B.

[16] I.e., *specular gloss* would be simply defined as reflected light whose intensity is greater than a given threshold, no matter the intensity of incident light. This basic conception of gloss is sometimes adopted, although such an approach is very much open to criticism (see (Chadwick) on (Shimomura)).

[17] It can be noticed that these conditions of assessment make *contrast gloss* prone to gloss consistency.

sharper on the pearl B, exhibiting a better *distinctness-of-reflected-image gloss*[18]. As well, figure 4 emphasizes the difference between d*istinctness-of-reflected-image gloss* and *specular gloss*: pearl B has a higher d*istinctness-of-reflected-image gloss* than pearl A, yet pearl A has a higher *specular gloss* than pearl B[19].

A numerical value corresponding to DOI could be the first derivative of the specular reflected intensity with respect to the angle of reflected light. On an image, it can be related to the magnitude of the gradient of the intensities around the specular reflectance zones.

Absence-of-bloom gloss: *Absence-of-bloom gloss* is the perceived brightness of a surface due to the absence of haze around specular reflected high-lights. As an illustration, in figure 5, pearl B has almost no haze compared to the two other pearls; as such, it can be said to have a better *absence-of-bloom gloss*. *Absence-of-bloom gloss* and *distinctness-of-reflected-image gloss* both relate to quality of reflected image. However, they account for two different aspects of it. The former accounts for how well the reflected image preserves intensity amplitudes around the specular angle, while the latter accounts for how well the reflected image preserves edge sharpness. On figure 5, pearl A and B both exhibit a high *distinctness-of-image gloss* compared to pearl C[20], yet pearl B has better *absence-of-bloom gloss* than pearl A[21].

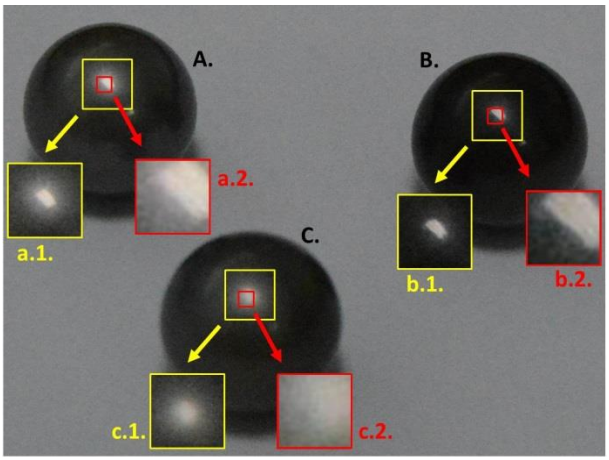


Figure 5: Pearls of different *absence-of-bloom gloss*. Pearl A and C have a lower *absence-of-bloom gloss* than pearl B.

Numerical values corresponding to *absence-of-bloom gloss* could be the ratio of the surface of haze to the surface of specular reflectance it surrounds. The difficulty is however to determine the exact surface of haze automatically. This feature could be learned in a supervised mode.

Absence-of-surface-texture gloss: *Absence-of-surface-texture gloss* is the perceived brightness of a surface due to the absence of interference between the reflected image and the image of the surface itself. Figure 6 illustrates the two images that can be formed out of a single pearl when its surface exhibits irregularities. Figure 6A focuses on reflected image while figure 6B focuses on the surface texture. Because the eyes can switch from one view to another, it creates an effect lessening the glossy appearance of the pearl.

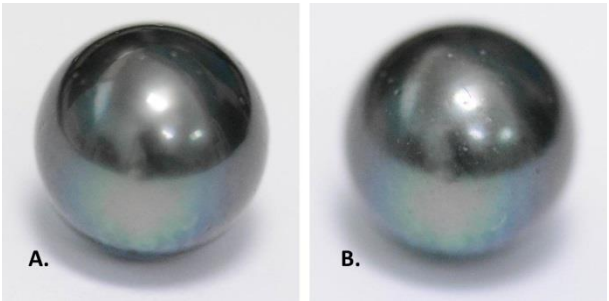


Figure 6: Pictures of a same pearl taken with two different focal distances (A focuses on the reflected image; B focuses on the surface of the pearl).

Absence-of-surface-texture gloss seems related to the quality of the surface, which is another criterion used by the experts to assess the quality of pearls, so, numerical values corresponding to *absence-of-surface-texture gloss* may be derived directly from a measure of the quality of the surface.

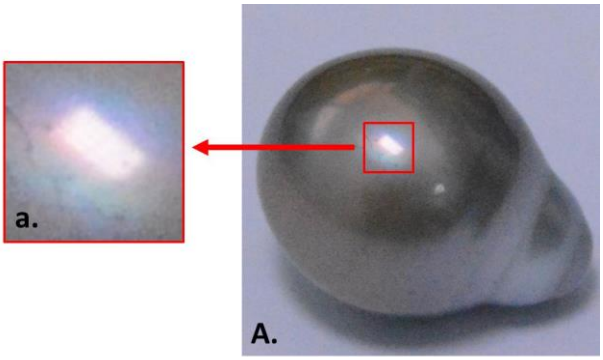


Figure 7: Some pearls exhibit colour changes around the specular angle, due to iridescence.

[18] An edge is visible between the left and the right parts of b., but not of a.

[19] Maximal intensity values of the image in an HSV colour space: 87.84% for the left pearl; 82.75% for the right pearl.

[20] An edge is visible between the left and the right parts of a.2. and b.2., but not of c.2.

[21] The difference of intensity between the left and the right parts of a.2. is lower than of b.2., but the edge between these parts is not sharper in b.2. than in a.2.

Iridescence: Since changes in wavelength due to specular reflection are negligible for dielectric materials, specular reflectance is usually studied as being achromatic, while colour is related to diffuse reflectance only. However, it has been seen that nacre exhibits iridescence [7], yet iridescence is a peculiar case since it takes form through colour changes, but is due to specular reflection. Figure 7 illustrates this aspect, with a zoom on the specular reflectance area, where parallel elongated coloured zones are visible, like superimposed. In some cases, rotating the pearl makes these zones rotate in the same direction. This anisotropic behaviour is observable on ringed pearls, like the one on figure 7.

Numerical values corresponding to *iridescence* could be the variance of chromaticity on the zone of the surface covered by the specular reflectance and surrounding haze.

Deep reflectance: Figure 8 shows the difference of reflectance between a fake pearl made of plastic (A) and a cultured pearl (B). Both pictures are taken in the exact same conditions. The fake pearl exhibits only a white reflectance zone while the cultured one exhibits both white and yellow reflectance zones. The reflectance split on the cultured pearl is not clearly visible to the naked eye, which only perceives an effect of 'deep' reflection when the pearl is rotated. Taking a picture of the pearl using a very short exposure time helps making it visible. Indeed, if exposure time is too long, the two reflectance zones merge in a single, larger reflectance zone, corresponding to the observation made with naked eye. Contrary to iridescence, where colours vary with the illumination and observation angles, the secondary zone here stays yellow regardless of the illumination or observation angles. A probable explanation is that, since conchiolin contains pigments, the different layers constituting nacre act like an absorbing filter: only part of the incident spectrum is specularly reflected from the deeper layers, the rest being absorbed by the successive layers of conchiolin. Figure 9 shows the same pearl as the one shown figure 7. The picture however is taken with a very short exposure time. One can observe both iridescence (b.3 to b.5) and deep reflectance (b.1).

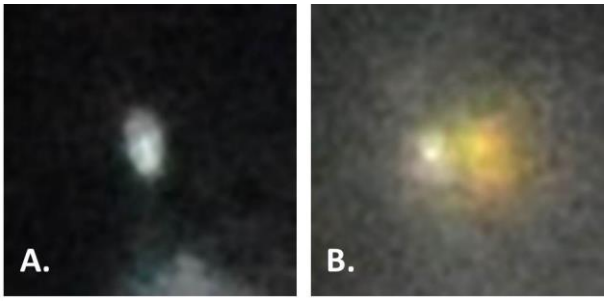


Figure 8: Detail of reflectance on two pearls (on the left, an artificial pearl; on the right, a real pearl).

Numerical values corresponding to *deep reflectance* could be computed using pictures taken using a very short exposure time. The ratio of the intensity of the deep reflectance point to the intensity of surface reflectance point could be a candidate. Furthermore, it might be that the distance between these two points has some impact.

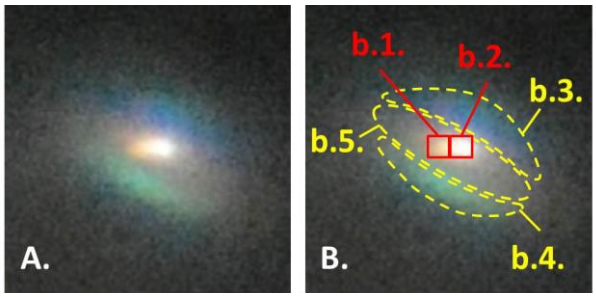


Figure 9: Difference between iridescence and deep reflectance.

5 Conclusion

In this paper, we present eight aspects of lustre which may be used to design feature vectors in a machine learning perspective for automatically assessing pearls' lustre. Each of these aspects is explained and illustrated and the distinctions between aspects are emphasized. Furthermore, with the goal of designing logical descriptors of lustre extractable from images of pearls, ways to obtain numerical values corresponding to these aspects are discussed. We have now to develop a protocol for images acquisition that ensures reproducibility and estimate the minimal number of samples needed for a machine learning process.

In addition to automatizing pearl's lustre assessment, machine learning could allow, by looking at the weights distribution yielded by the fitting process, to investigate the actual impact of each aspect of lustre on pearl's lustre assessment, to better understand how human assessment of lustre works.

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